Many different uniformity numbers of Yorioka ideals

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Definition (Yorioka 2002)

For $f : \omega \to \omega$ increasing, define

$$\mathcal{I}_f := \bigcup \{\mathcal{J}_g : g \gg f\}$$

where $f \ll g$ iff g dominates $\{f \circ \mathrm{id}^n : n < \omega\}$ $(\mathrm{id}^n(i) = i^n)$.

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Each $\mathcal{I}_f$ is a σ-ideal and $\mathcal{SN} \subseteq \mathcal{I}_f \subseteq \mathcal{N}$.

Many cardinals

Theorem (Kamo & Osuga 2014)

If κ is an uncountable regular cardinal then there is a ccc poset forcing that $\{\mathrm{cov}(\mathcal{I}_{f_\alpha}) : \alpha < \kappa\}$ is pairwise different for some $\{f_\alpha : \alpha < \kappa\}$.

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Problem

Is it consistent that there are infinitely many

1. $\mathrm{add}(\mathcal{I}_f)$?
2. $\mathrm{non}(\mathcal{I}_f)$?
3. $\mathrm{cof}(\mathcal{I}_f)$?

Localization and anti-localization cardinals

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Theorem (Bartoszyński 1980's)

1. *If* $h \to \infty$ *then* $\mathfrak{b}^{\mathrm{Lc}}_{\omega,h} = \mathrm{add}(\mathcal{N})$ *and* $\mathfrak{d}^{\mathrm{Lc}}_{\omega,h} = \mathrm{cof}(\mathcal{N})$.

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Theorem (Kellner & Shelah 2012)

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Is it consistent that there are infinitely many pairwise different $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$?

Kamo-Osuga inequalities

Lemma (Kamo & Osuga 2014)

If $c \geq^* 2$ *and* $\forall^\infty k\left(\sum_{i\leq k}\log_2 c(i) \leq g(k)\right)$ *then* $\mathfrak{b}^{\mathrm{aLc}}_{c,1} \leq \mathrm{cov}(\mathcal{J}_g)$ *and* $\mathrm{non}(\mathcal{J}_g) \leq \mathfrak{d}^{\mathrm{aLc}}_{c,1}$.

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If g *is monotonically increasing and* $\forall^\infty k \left(2^{g\left(\sum_{i\leq k} h(i)-1\right)} \leq c(k)\right)$ *then* $\mathrm{cov}(\mathcal{J}_g) \leq \mathfrak{b}^{\mathrm{aLc}}_{c,h}$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{c,h} \leq \mathrm{non}(\mathcal{I}_g)$.

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Remark

If $f \ll g$ then $\mathcal{J}_g \subseteq \mathcal{I}_f \subseteq \mathcal{J}_f$, so $\mathrm{cov}(\mathcal{J}_f) \leq \mathrm{cov}(\mathcal{I}_f) \leq \mathrm{cov}(\mathcal{J}_g)$ and $\mathrm{non}(\mathcal{J}_g) \leq \mathrm{non}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{J}_f)$.

Aiming for many $\mathfrak{d}^{\mathrm{aLc}}_{c,h}$ and $\mathrm{non}(\mathcal{I}_f)$

Forcing to increase $\mathfrak{d}^{\mathrm{aLc}}_{c,h}$: assume $1 \leq h < c$ and $\limsup_n h(n) = \infty$.

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E.g., $\mathrm{nor}^{-}(M) \geq 1$ iff $\bigcup M = [c(n)]^{\leq h(n)}$; $\mathrm{nor}^{-}(\{a\}) = 0$.

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E.g., $\mathrm{nor}^-(M) \geq 1$ iff $\bigcup M = [c(n)]^{\leq h(n)}$; $\mathrm{nor}^-(\{a\}) = 0$.

2. Define

$$\mathbb{Q}^-_{c,h} := \left\{p \in \prod_{n<\omega} \mathbf{K}(n) : \limsup_n \{\mathrm{nor}^-(p(n))\} = \infty\right\}$$

ordered by $q \leq p$ iff $\forall n(q(n) \subseteq p(n))$.

Some properties of the forcing

$\mathbb{Q}^{-}_{c,h}$ adds a generic real $\varphi^* \in \mathcal{S}(c,h)$ such that $\forall x \in V \cap \prod c (x \in^\infty \varphi^*)$.

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Task

For which $b, g \in \omega^\omega$ the cardinal $\mathfrak{d}^{\mathrm{aLc}}_{b,g}$ is not modified by $\mathbb{Q}^-_{c,h}$?

Modify the norm

Fix $d : \omega \to \omega \setminus 2$ such that $\lim_n \frac{1}{d(n)} \log_{d(n)} h(n) = \infty$.

Definition

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$$\mathrm{nor}(M) := \frac{1}{d(n)} \log_{d(n)}(\mathrm{nor}^-(M) + 1).$$

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Lemma (Bigness)

Whenever $M \in \mathbf{K}(n)$ and $f : M \to d(n)$, there is some $M^* \subseteq M$ such that $f \restriction M^*$ is constant and $\mathrm{nor}(M^*) \geq \mathrm{nor}(M) - \frac{1}{d(n)}$.

Main Lemma

The poset $\mathbb{Q}^d_{c,h}$ forces $\forall x \in V[G] \cap \prod a \exists \varphi \in V \cap \mathcal{S}(a,e)(x \in^* \varphi)$ whenever

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So $\mathbb{Q}^d_{c,h}$ increases $\mathfrak{d}^{\mathrm{aLc}}_{c,h}$ while preserving $\mathfrak{d}^{\mathrm{Lc}}_{a,e}$.

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Lemma

If $\forall^\infty i$: $b(i)^{g(i)} \leq a(i)$ *and* $e(i) < \frac{b(i)}{g(i)}$, *then* $\mathfrak{b}^{\mathrm{Lc}}_{a,e} \leq \mathfrak{b}^{\mathrm{aLc}}_{b,g}$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{b,g} \leq \mathfrak{d}^{\mathrm{Lc}}_{a,e}$.

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Theorem (Klausner & M.)

Assume CH. Let K be a set of cardinals, $|K| \leq \aleph_1$, such that $\kappa^{\aleph_0} = \kappa$ for any $\kappa \in K$.

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1. $\mathfrak{d}^{\mathrm{aLc}}_{c_\kappa, h_\kappa} \leq \mathrm{non}(\mathcal{I}_{f_\kappa}) \leq \mathfrak{d}^{\mathrm{Lc}}_{a_\kappa, d_\kappa}$ *for any $\kappa \in K$, and*

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2. *there is a proper ω^ω-bounding $\aleph_2$-cc poset $\mathbb{Q}$ forcing* $\mathfrak{d}^{\mathrm{aLc}}_{c_\kappa, h_\kappa} = \mathrm{non}(\mathcal{I}_{f_\kappa}) = \mathfrak{d}^{\mathrm{Lc}}_{a_\kappa, d_\kappa} = \kappa$ *for any $\kappa \in K$.*

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2. *there is a proper ω^ω-bounding $\aleph_2$-cc poset $\mathbb{Q}$ forcing* $\mathfrak{d}^{\mathrm{aLc}}_{c_\kappa, h_\kappa} = \mathrm{non}(\mathcal{I}_{f_\kappa}) = \mathfrak{d}^{\mathrm{Lc}}_{a_\kappa, d_\kappa} = \kappa$ *for any $\kappa \in K$.*

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Main result

Theorem (Klausner & M.)

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The poset $\mathbb{Q}$ is the CS product of $\langle \mathbb{Q}_i : i \in I\rangle$ where

1. $|I| = \sum_{\kappa\in K} \kappa$,
2. $\langle I_\kappa : \kappa \in I\rangle$ is a partition of I, $|I_\kappa| = \kappa$, and
3. $\mathbb{Q}_i := \mathbb{Q}^{d_\kappa}_{c_\kappa,h_\kappa}$ for any $i \in I_\kappa$.

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Lemma

$\mathfrak{b}^{\mathrm{aLc}}_{b,g} \leq \mathrm{cov}(\mathcal{I}_f)$ *and* $\mathrm{non}(\mathcal{I}_f) \leq \mathfrak{d}^{\mathrm{aLc}}_{b,g}$ *whenever*

1. $\langle J_n : n < \omega\rangle$ *is an interval partition of* ω, $|J_n| = g(n)$,
2. $f_{b,g}(k) := \sum_{i \leq n} \log_2 b(i)$ *when* $k \in J_n$, *and*
3. $f_{b,g} \leq^* f \circ \mathrm{id}^m$ *for some* $m \geq 1$.

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Lemma

$\mathrm{cov}(\mathcal{I}_f) \leq \mathfrak{b}^{\mathrm{aLc}}_{c,h}$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{c,h} \leq \mathrm{non}(\mathcal{I}_f)$ *whenever*

1. $\langle I_n : n<\omega\rangle$ *is an interval partition of* ω, $|I_n| = h(n)$,
2. $g_{c,h}(k) := \log_2 c(n)$ *when* $k \in I_n$, *and*
3. $f \ll g_{c,h}$.

Open questions

Question 1

Is it consistent that there are $\mathfrak{c}$-many pairwise different cardinals of the form

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Same problem with cof.