Some applications of iterated ultrapowers in countably compact groups

Ulises Ariet RAMOS GARCÍA

National Autonomous University of Mexico, Morelia
ariet@matmor.unam.mx

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Contenido

Versions of compatness

Given an ultrafilter p on ω, a sequence $\langle x_n : n \in \omega \rangle$ and a point x contained in a topological space X we say that $x = p\text{-}\lim x_n$ if $\{n \in \omega : x_n \in U\} \in p$ for every neighbourhood $U \subseteq X$ of x.

Definition

Let X be a topological space and let p be an ultrafilter on ω.

- X is compact if every open cover of X has a finite subcover.
- X is p-compact if for every sequence $\langle x_n : n \in \omega \rangle \subseteq X$ there is a point $x \in X$ such that $x = p\text{-}\lim x_n$.
- X is countably compact if every countable open cover of X has a finite subcover.
  - iff every infinite set has an accumulation point.
  - iff for every sequence $\langle x_n : n \in \omega \rangle \subseteq X$ there is a point $x \in X$ and an ultrafilter p on ω such that $x = p\text{-}\lim x_n$.
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- (Tychonoff '30/'35) Any product of compact spaces is compact.
- (Ginsburg-Saks '75) Any product of p-compact spaces is p-compact.
- (Tereska '52, Novák '53) There are countably compact spaces whose square is not even pseudo-compact.
- (Comfort-Ross '66) Any product of pseudo-compact topological groups is pseudo-compact.

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Are there countably compact groups G and H such that $G \times H$ is not countably compact?

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- (van Douwen '80) (MA) There are two countably compact subgroups of $2^{\mathfrak{c}}$ whose product is not countably compact.
- (van Mill-Hart '91) (MA_{ctble}) There is a countably compact group whose square is not countably compact.
- (Tomita '99) (MA_{ctble}) There is a group whose square is countably compact but the cube is not.
- (van Douwen '80) Every countably compact Boolean group without (non-trivial) convergent sequences contains two countably compact subgroups whose product is not countably compact.
- (Hajnal-Juhász '76) (CH) There is a countably compact Boolean group without convergent sequences.

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Is there a countably compact Boolean group without non-trivial convergent sequences?

- (Kuz'minov '58) Every compact topological group contains a non-trivial convergent sequence.
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- ...
- (García Ferreira-Tomita-Watson '05) Yes assuming the existence of a selective ultrafilter. In fact, such group is p-compact.
- (Szeptycki-Tomita '09) Yes in the random real model.

All of these constructions describe subgroups of $2^{\mathfrak{c}}$.

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Contenido

Main result

Theorem

There is (in ZFC) a countably compact Boolean group (subgroup of $2^{\mathfrak{c}}$) without non-trivial convergent sequences.

Bohr group topologies

The Bohr topology on a group G is the weakest group topology making every homomorphism $\Phi \in \mathrm{Hom}(G, \mathbb{T})$ continuous. We let $(G, \tau_{\mathrm{Bohr}})$ denote G equipped with the Bohr topology.

- (Folklore) If G is an Abelian group, then $(G, \tau_{\mathrm{Bohr}})$ is homeomorphic (and isomorphic) to a subgroup of $\mathbb{T}^{\mathrm{Hom}(G,\mathbb{T})}$ via the evaluation mapping.

Theorem

Let G be a (infinite) countable Abelian group. Then $(G, \tau_{\mathrm{Bohr}})$ has no (non-trivial) convergent sequences.

- (Folklore) Let X be a countably compact regular space without (non-trivial) convergent sequences. Then every infinite subset of X has at least $\mathfrak{c}$ accumulation points.

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Extensions of homomorphisms to ultrapowers

Given a group G and an ultrafilter $p \in \omega^*$, denote by

$$\mathrm{ult}_p(G) = (G)^\omega / \equiv_p \text{, where } f \equiv_p g \text{ iff } \{n : f(n) = g(n)\} \in p.$$

- By Łós theorem, $\mathrm{ult}_p(G)$ is a group with the same first order properties as G.
- There is a natural embedding of G into $\mathrm{ult}_p(G)$ sending each $g \in G$ to the equivalence class of the constant function with value g. We shall therefore consider G as a subgroup of $\mathrm{ult}_p(G)$.

Every $\Phi \in \mathrm{Hom}(G, \mathbb{T})$ naturally extends to a homomorphism $\overline{\Phi} \in (\mathrm{ult}_p(G), \mathbb{T})$ by letting

$$\overline{\Phi}([f]) = p\text{-}\lim_{n\in\omega} \Phi(f(n)).$$

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Iterated ultrapowers

The process can, of course, be iterated: given a group G, an ultrafilter $p \in \omega^*$ and $0 < \alpha \leqslant \omega_1$ let

$$\mathrm{ult}_p^{\alpha}(G) = \mathrm{ult}_p(ult_p^{\beta}(G)) \text{ if } \alpha = \beta + 1,$$

and

$$\mathrm{ult}_p^{\alpha}(G) = \bigcup_{0<\beta<\alpha} \mathrm{ult}_p^{\beta}(G) \text{ if } \alpha \text{ is limit.}$$

The group that will be relevant for us is the group $\mathrm{ult}_p^{\omega_1}(G)$, endowed with the topology $\tau_{\overline{\mathrm{Bohr}}}$ induced by the homomorphisms in $\Phi \in \mathrm{Hom}(G, \mathbb{T})$ extended recursively all the way to $\mathrm{ult}_p^{\omega_1}(G)$ by the same formula as before:

$$\overline{\Phi}([f]) = p\text{-}\lim_{n\in\omega} \overline{\Phi}(f(n)).$$

The (iterated) ultrapower with this topology is usually not Hausdorff, so we identify the inseparable functions and denote by $\mathrm{Ult}_p^{\omega_1}(G)$ this quotient.

Fact

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Theorem

There is (in ZFC) a countably compact Boolean group (subgroup of $2^{\mathfrak{c}}$) without non-trivial convergent sequences.

Fix an ultrafilter $p \in \omega^*$ and consider $\mathrm{Ult}_p^{\omega_1}(G)$. There is a problem:

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For every countable Abelian group G there is an ultrafilter $p \in \omega^*$ s.t. $\mathrm{Ult}_p(G)$ has a non-trivial convergent sequences.

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The plan works for selective ultrafilters and for $([\omega]^{<\omega}, \triangle)$

Proposition

p is selective iff for every p-independent set $\{f_n \colon n \in \omega\}$ of functions $f_n \colon \omega \to [\omega]^{<\omega}$, there is a sequence $\langle U_n \colon n \in \omega \rangle \subseteq p$ such that $f_n \restriction U_n$ is one-to-one for every $n \in \omega$ and $\{f_n(m) \colon n \in \omega \text{ and } m \in U_n\}$ is linearly independent.

Given a non-empty set I, we shall call a set $\{f_i \colon i \in I\}$ of functions $f_i \colon \omega \to [\omega]^{<\omega}$ *p-independent* if

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for every non-empty finite set $E \subset I$ and every $a \in [\omega]^{<\omega}$.

Corollary

If p is selective, then $\mathrm{Ult}_p^{\omega_1}([\omega]^{<\omega})$ is a Hausdorff p-compact topological group without (non-trvial) convergent sequences.

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Proposition

The group topology $\tau_{\overline{Bohr}}$ contains no non-trivial convergent sequences iff $\forall D \in [\mathfrak{c}]^\omega$ suitably closed $\forall X \in [D]^\omega$ $\exists \Psi \in Hom([D]^{<\omega}, 2)$ s.t.

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Final remarks and questions

Theorem

For every $n \in \omega$ there is a group G such that G^n is countably compact while G^{n+1} is not.

Questions

1. Is there a countably compact group G without convergent sequences which is not a torsion group, *i.e.*, contains a copy of $\mathbb{Z}$?
2. (Wallace '55) Is there a Hausdorff countably compact semigroup with both-sided cancellation which is not a topological group?

Yes to 1 implies Yes to 2.

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For every $n \in \omega$ there is a group G such that G^n is countably compact while G^{n+1} is not.

Questions

1. Is there a countably compact group G without convergent sequences which is not a torsion group, *i.e.*, contains a copy of $\mathbb{Z}$?
2. (Wallace '55) Is there a Hausdorff countably compact semigroup with both-sided cancellation which is not a topological group?

Yes to 1 implies Yes to 2.

Questions

- Is there consistently a countably compact group G without convergent sequences of weight less than $\mathfrak{c}$?
- Is there (in ZFC) a p-compact group G without convergent sequences?
- Is there (in ZFC) a $p \in \omega^*$ and a group G s.t. $\mathrm{Ult}_p(G)$ has no (non-trivial) convergent sequences?

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Thank you for your attention!