Some problems related to internal approachability

Sean Cox

Virginia Commonwealth University
scox9@vcu.edu
skolemhull.wordpress.com

Axiomatic set theory and its applications
RIMS
November 2018

Outline

Internal approachability and related concepts

$$P^*_{\omega_2}(V) := \{W \ : \ |W| = \omega_1 \subset W\}$$

Internal approachability and related concepts

$$P^*_{\omega_2}(V) := \{ W \ : \ |W| = \omega_1 \subset W \}$$

Filtration of $\boldsymbol{W}$: a $\subseteq$-increasing and continuous sequence $\vec{N} = \langle N_i \ : \ i < \omega_1 \rangle$ with each $|N_i| = \omega$, and $W = \bigcup_{i<\omega_1} N_i$.

Internal approachability and related concepts

$$P^*_{\omega_2}(V) := \{W \ : \ |W| = \omega_1 \subset W\}$$

Filtration of *W*: a $\subseteq$-increasing and continuous sequence $\vec{N} = \langle N_i \ : \ i < \omega_1 \rangle$ with each $|N_i| = \omega$, and $W = \bigcup_{i<\omega_1} N_i$.

IA := class of $W \in P^*_{\omega_2}(V)$ s.t. there exists a filtration of W, all of whose proper initial segments are elements of W.

Internal approachability and related concepts

$$P^*_{\omega_2}(V) := \{W \ : \ |W| = \omega_1 \subset W\}$$

Filtration of $\boldsymbol{W}$: a $\subseteq$-increasing and continuous sequence $\vec{N} = \langle N_i \ : \ i < \omega_1 \rangle$ with each $|N_i| = \omega$, and $W = \bigcup_{i<\omega_1} N_i$.

IA := class of $W \in P^*_{\omega_2}(V)$ s.t. there exists a filtration of W, all of whose proper initial segments are elements of W.

Definition (Foreman-Todorcevic)

IC := $\{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ contains a club in $[W]^\omega\}$
IS := $\{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ is stationary in $[W]^\omega\}$
IU := $\{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ is $\subseteq$-cofinal in $[W]^\omega\}$

Internal approachability and related concepts

$$P^*_{\omega_2}(V) := \{W \ : \ |W| = \omega_1 \subset W\}$$

Filtration of $\boldsymbol{W}$: a $\subseteq$-increasing and continuous sequence $\vec{N} = \langle N_i \ : \ i < \omega_1 \rangle$ with each $|N_i| = \omega$, and $W = \bigcup_{i<\omega_1} N_i$.

IA $:=$ class of $W \in P^*_{\omega_2}(V)$ s.t. there exists a filtration of W, all of whose proper initial segments are elements of W.

Definition (Foreman-Todorcevic)

IC $:= \{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ contains a club in $[W]^\omega\}$
IS $:= \{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ is stationary in $[W]^\omega\}$
IU $:= \{W \in P^*_{\omega_2}(V) \ : \ W \cap [W]^\omega$ is $\subseteq$-cofinal in $[W]^\omega\}$

Easily: IA $\subseteq$ IC $\subseteq$ IS $\subseteq$ IU.

When are the inclusions strict?

Fact (Foreman-Todorcevic)

Continuum Hypothesis implies $IA =^* IC =^* IS =^* IU$.

When are the inclusions strict?

Fact (Foreman-Todorcevic)

Continuum Hypothesis implies $IA =^* IC =^* IS =^* IU$.

Lemma

Assume $2^{\omega_1} = \omega_2$. TFAE:

1. *The* **Approachability Property** *fails at ω_2 (i.e. $\omega_2 \notin I[\omega_2]$)*
2. **At least one** *of the three inclusions $IA \subseteq IC \subseteq IS \subseteq IU$ is strict (mod clubs) in $P_{\omega_2}(H_{\omega_2})$.*

When are the inclusions strict?

Fact (Foreman-Todorcevic)

Continuum Hypothesis implies $IA =^* IC =^* IS =^* IU$.

Lemma

Assume $2^{\omega_1} = \omega_2$. *TFAE:*

1. *The* **Approachability Property** *fails at* ω_2 *(i.e.* $\omega_2 \notin I[\omega_2]$*)*
2. **At least one** *of the three inclusions* $IA \subseteq IC \subseteq IS \subseteq IU$ *is strict (mod clubs) in* $P_{\omega_2}(H_{\omega_2})$.

Theorem

IA	$\subseteq$	IC	$\subseteq$	IS	$\subseteq$	IU
	*strict under PFA (***Krueger***)*		*strict under* PFA^+*, MM, but not PFA (***Cox***)*		*strict under MM (***Krueger***), but not under* PFA^{++} *(***Cox***)*	

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W \ : \ |W| = \omega_1 \subset W\}$.

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W \ : \ |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP}_{\boldsymbol{\Delta}}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary} \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W \ : \ |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP}_{\boldsymbol{\Delta}}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$$\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}} \implies \mathrm{RP}_{\mathrm{IU}}\big(\iff \textbf{Axiom R}\big) \implies \textbf{WRP}$$

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W \ : \ |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP_\Delta}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}} \implies \mathrm{RP}_{\mathrm{IU}} \big(\iff$ **Axiom R**$\big) \implies$ **WRP**

Q: Which can be reversed?

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W \ : \ |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP}_{\boldsymbol{\Delta}}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$$\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}} \implies \mathrm{RP}_{\mathrm{IU}} \big(\iff \textbf{Axiom R}\big) \implies \textbf{WRP}$$

Q: Which can be reversed?

1. (Beaudoin 1980s) Can $\mathrm{RP}_{\mathrm{IU}} \implies$ WRP be reversed? (open)
    - Related, and more recent: Does WRP imply *Fodor-type Reflection Principle (FRP)*?

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W : |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP_\Delta}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$\mathrm{RP_{IA}} \implies \mathrm{RP_{IC}} \implies \mathrm{RP_{IS}} \implies \mathrm{RP_{IU}} (\iff$ **Axiom R**$) \implies$ **WRP**

Q: Which can be reversed?

1. (Beaudoin 1980s) Can $\mathrm{RP_{IU}} \implies$ WRP be reversed? (open)
   - Related, and more recent: Does WRP imply *Fodor-type Reflection Principle (FRP)*?
2. (Foreman-Todorcevic 2000s) Can $\mathrm{RP_{IA}} \implies$ WRP be reversed?

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W : |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP}_{\mathbf{\Delta}}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}} \implies \mathrm{RP}_{\mathrm{IU}}(\iff$ **Axiom R**$) \implies$ **WRP**

Q: Which can be reversed?

1. (Beaudoin 1980s) Can $\mathrm{RP}_{\mathrm{IU}} \implies$ WRP be reversed? (open)
   - Related, and more recent: Does WRP imply *Fodor-type Reflection Principle (FRP)*?
2. (Foreman-Todorcevic 2000s) Can $\mathrm{RP}_{\mathrm{IA}} \implies$ WRP be reversed? **No (Krueger; strengthened and simplified by Cox)**

Reflection to the Foreman-Todorcevic classes

Let Δ denote any subclass of $\{W : |W| = \omega_1 \subset W\}$.

Definition (Foreman-Todorcevic)

$\mathbf{RP}_{\boldsymbol{\Delta}}$: "Every stat. subset of $[H_\theta]^\omega$ reflects to a member of Δ"

$$(\forall S \subseteq [H_\theta]^\omega)\Big(S \text{ stationary } \implies (\exists W \in \Delta)(S \cap [W]^\omega \text{ is stationary})\Big).$$

$\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}} \implies \mathrm{RP}_{\mathrm{IU}}(\iff$ **Axiom R**$) \implies$ **WRP**

Q: Which can be reversed?

1. (Beaudoin 1980s) Can $\mathrm{RP}_{\mathrm{IU}} \implies$ WRP be reversed? (open)
   - Related, and more recent: Does WRP imply *Fodor-type Reflection Principle (FRP)*?
2. (Foreman-Todorcevic 2000s) Can $\mathrm{RP}_{\mathrm{IA}} \implies$ WRP be reversed? **No (Krueger; strengthened and simplified by Cox)**
3. (Krueger 2000s) Can $\mathrm{RP}_{\mathrm{IS}} \implies$ WRP be reversed? (open)

Outline

Regarding $\mathsf{RP}_{\mathsf{IA}} \implies \mathsf{RP}_{\mathsf{IC}} \implies \mathsf{RP}_{\mathsf{IS}}$

Regarding $\mathsf{RP}_{\mathsf{IA}} \implies \mathsf{RP}_{\mathsf{IC}} \implies \mathsf{RP}_{\mathsf{IS}}$

Let $\Delta \subseteq \{W \ : \ |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_Δ.

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W : |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_{Δ}.

- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W \ : \ |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_{Δ}.

- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"
- (Larson) DRP strictly stronger than $\mathbf{RP}^{\omega_1}$

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W \ : \ |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_{Δ}.

- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"
- (Larson) DRP strictly stronger than $\mathbf{RP}^{\omega_1}$
- $\mathrm{PFA}^{++} \implies \mathrm{DRP}_{\mathrm{IA}}$

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W : |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_Δ.

- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"
- (Larson) DRP strictly stronger than $\mathbf{RP}^{\omega_1}$
- $\mathrm{PFA}^{++} \implies \mathrm{DRP}_{\mathrm{IA}}$

Theorem (Cox—strengthening/simplifying a theorem of Krueger)

DRP_{IS} does not imply RP_{IC}. (In fact, even Fuchino-Usuba's $\mathbf{DRP}_{\mathbf{internal}}$ *does not imply RP_{IC}).*

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W : |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\boldsymbol{\Delta}}$ is a highly simultaneous version of RP_Δ.

- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"
- (Larson) DRP strictly stronger than $\mathbf{RP}^{\omega_1}$
- $\mathrm{PFA}^{++} \implies \mathrm{DRP}_{\mathrm{IA}}$

Theorem (Cox—strengthening/simplifying a theorem of Krueger)

DRP_{IS} does not imply RP_{IC}. (In fact, even Fuchino-Usuba's $\mathbf{DRP}_{\mathbf{internal}}$ *does not imply RP_{IC}).*
DRP_{IC} does not imply RP_{IA}.

Regarding $\mathrm{RP}_{\mathrm{IA}} \implies \mathrm{RP}_{\mathrm{IC}} \implies \mathrm{RP}_{\mathrm{IS}}$

Let $\Delta \subseteq \{W \ : \ |W| = \omega_1 \subset W\}$

Definition (Diagonal Reflection Principle (C.))

$\mathbf{DRP}_{\mathbf{\Delta}}$ is a highly simultaneous version of RP_{Δ}.

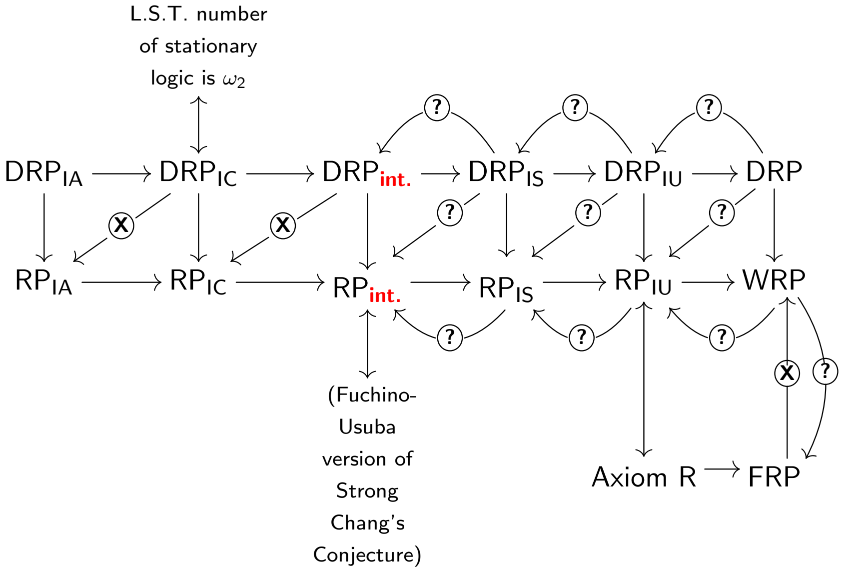
- (Fuchino) $\mathrm{DRP}_{\mathrm{IC}} \iff$ "LST(**stationary logic**) $= \omega_2$"
- (Larson) DRP strictly stronger than $\mathbf{RP}^{\omega_1}$
- $\mathrm{PFA}^{++} \implies \mathrm{DRP}_{\mathrm{IA}}$

Theorem (Cox—strengthening/simplifying a theorem of Krueger)

DRP_{IS} does not imply RP_{IC}. (In fact, even Fuchino-Usuba's $\mathbf{DRP}_{\mathbf{internal}}$ *does not imply RP_{IC}).*
DRP_{IC} does not imply RP_{IA}.

Either can be simply forced over an arbitrary model of PFA^{++}.

Summary of known relationships



Outline

Recall . . .

Theorem (Cox)

$DRP_{IS} \;\not\Rightarrow\; RP_{IC}$.

Recall . . .

Theorem (Cox)

$DRP_{IS} \not\Rightarrow RP_{IC}$.

Proof has 2 parts:

- $\mathrm{FA}^{++}\big($"proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$"$\big)$ implies $\mathrm{DRP}_{\mathrm{IS}}$.
- Forcing axiom from first bullet preserved by killing $\mathrm{RP}_{\mathrm{IC}}$.

Recall . . .

Theorem (Cox)

$DRP_{IS} \not\Rightarrow RP_{IC}$.

Proof has 2 parts:

- FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$”) implies $\mathrm{DRP}_{\mathrm{IS}}$.
- Forcing axiom from first bullet preserved by killing $\mathrm{RP}_{\mathrm{IC}}$.

(Proof that $\mathrm{DRP}_{\mathrm{IC}} \not\Rightarrow \mathrm{RP}_{\mathrm{IA}}$ has similar outline)

Recall . . .

Theorem (Cox)

$DRP_{IS} \not\Rightarrow RP_{IC}$.

Proof has 2 parts:

- FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$”) implies $\mathsf{DRP}_{\mathsf{IS}}$.
- Forcing axiom from first bullet preserved by killing $\mathsf{RP}_{\mathsf{IC}}$.

(Proof that $\mathsf{DRP}_{\mathsf{IC}} \not\Rightarrow \mathsf{RP}_{\mathsf{IA}}$ has similar outline)

Suggests a natural attempt to prove $\mathsf{DRP}_{\mathsf{IU}} \not\Rightarrow \mathsf{RP}_{\mathsf{IS}}$:

Recall . . .

Theorem (Cox)

$DRP_{IS} \not\Rightarrow RP_{IC}$.

Proof has 2 parts:

- FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$") implies $\mathrm{DRP}_{\mathrm{IS}}$.
- Forcing axiom from first bullet preserved by killing $\mathrm{RP}_{\mathrm{IC}}$.

(Proof that $\mathrm{DRP}_{\mathrm{IC}} \not\Rightarrow \mathrm{RP}_{\mathrm{IA}}$ has similar outline)

Suggests a natural attempt to prove $\mathrm{DRP}_{\mathrm{IU}} \not\Rightarrow \mathrm{RP}_{\mathrm{IS}}$:

- Does FA^{++}("stat-set-preserving posets that force $H^V_{\omega_2} \in \mathrm{IU} \setminus \mathrm{IS}$") imply $\mathrm{DRP}_{\mathrm{IU}}$?
- Is the forcing axiom from previous bullet preserved by the forcing to kill $\mathrm{RP}_{\mathrm{IS}}$?

Recall . . .

Theorem (Cox)

$DRP_{IS} \not\Rightarrow RP_{IC}$.

Proof has 2 parts:

- FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$") implies $\mathrm{DRP}_{\mathrm{IS}}$.
- Forcing axiom from first bullet preserved by killing $\mathrm{RP}_{\mathrm{IC}}$.

(Proof that $\mathrm{DRP}_{\mathrm{IC}} \not\Rightarrow \mathrm{RP}_{\mathrm{IA}}$ has similar outline)

Suggests a natural attempt to prove $\mathrm{DRP}_{\mathrm{IU}} \not\Rightarrow \mathrm{RP}_{\mathrm{IS}}$:

- Does FA^{++}("stat-set-preserving posets that force $H^V_{\omega_2} \in \mathrm{IU} \setminus \mathrm{IS}$") imply $\mathrm{DRP}_{\mathrm{IU}}$?
- Is the forcing axiom from previous bullet preserved by the forcing to kill $\mathrm{RP}_{\mathrm{IS}}$? **answer to this bullet is "yes". However . . .**

A no-go theorem

(from previous slide):

1. Does $\mathsf{FA}^{++}\big($"stat-set-preserving posets that force $H^V_{\omega_2} \in \mathsf{IU} \setminus \mathsf{IS}$"$\big)$ imply $\mathsf{DRP}_{\mathsf{IU}}$?
2. Is the forcing axiom from previous bullet preserved by the forcing to kill $\mathsf{RP}_{\mathsf{IS}}$?

A no-go theorem

(from previous slide):

1. Does
   FA^{++}(“stat-set-preserving posets that force $H^V_{\omega_2} \in \mathrm{IU} \setminus \mathrm{IS}$”)
   imply $\mathrm{DRP}_{\mathrm{IU}}$?
2. Is the forcing axiom from previous bullet preserved by the forcing to kill $\mathrm{RP}_{\mathrm{IS}}$?

Answer to 1st bullet is **no**:

Theorem (Cox)

*The forcing axiom from (1) does not even imply WRP(ω_2).
(this forcing axiom is preserved by the poset of Aspero et al that adds a nonreflecting stationary subset of $[\omega_2]^\omega$).*

Outline

Preserving forcing axioms via lifting embeddings

Assume Γ is "nice" class of posets.

Preserving forcing axioms via lifting embeddings

Assume Γ is "nice" class of posets.

Theorem (Forcing Axiom Preservation; see Thm 20 of Cox [1])

Assume $FA(\Gamma)$, *and* $\mathbb{P}$ *has the property: if* $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$, *then there exists a* $\mathbb{P} * \dot{\mathbb{Q}}$*-name* $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ *such that:*

- $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in \Gamma$;
- **Whenever** $j : V \to N$ *is a generic elementary embedding with* $crit(j) = \omega_2^V$, *and*
    - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in wfp(N)$, $|H_\theta^V|^N = \omega_1$, *and* $j \restriction H_\theta^V \in N$;
    - *in* N, *there exists a* $(V, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$*-generic object* $G * H * K$;

  then N *believes that* $j[G]$ *has a lower bound in* $j(\mathbb{P})$.

Then $V^{\mathbb{P}} \models FA(\Gamma)$.

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed **(Larson)**

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed (Larson)

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed (Larson)

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.
3. Then $\mathbb{P} * \dot{\mathbb{Q}} (* \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in \Gamma$ by "niceness" of Γ.

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed **(Larson)**

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
    - $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed (Larson)

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
   - $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
   - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed (Larson)

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\text{crit}(j) = \omega_2^V$ and:
   - $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in H_\theta^V \in \text{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
   - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$
5. Then $j[G] \in N$, and has size ω_1 there. By directed closure of $j(\mathbb{P})$ in N, $j[G]$ has a lower bound.

Example: Preservation of FA(Γ) by $< \omega_2$ directed closed **(Larson)**

Assume FA(Γ), and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset.
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\text{crit}(j) = \omega_2^V$ and:
    - $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}) \in H_\theta^V \in \text{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
    - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$
5. Then $j[G] \in N$, and has size ω_1 there. By directed closure of $j(\mathbb{P})$ in N, $j[G]$ has a lower bound.

So by the Preservation Theorem, FA(Γ) holds in $V[G]$.

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \mathsf{PFA}\big(= \mathsf{FA}(\text{proper})\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player **II**.

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \mathrm{PFA}\big(= \mathrm{FA}(\text{proper})\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player **II**. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \mathrm{PFA}\big(= \mathrm{FA}(\mathrm{proper})\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".

(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \mathrm{PFA}\big(= \mathrm{FA}(\mathrm{proper})\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".
(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

- Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \text{PFA}\big(= \text{FA(proper)}\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".
(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

- Assume $j : V \to N$ is such that $\text{crit}(j) = \omega_2^V$ and:
  - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in \text{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \mathrm{PFA}\big(= \mathrm{FA}(\text{proper})\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".
(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

- Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
  - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
  - In N, there is $G * H * K$ generic over V for $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$.

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \text{PFA}\big(= \text{FA(proper)}\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".

(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

- Assume $j : V \to N$ is such that $\text{crit}(j) = \omega_2^V$ and:
    - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in \text{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
    - In N, there is $G * H * K$ generic over V for $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$.
- Then $H_\theta^V[G * H * K]$, and hence N, have the ω_1-length play $\mathcal{P}$ described above.

Example: preservation of PFA by $\omega_1 + 1$ **operationally closed** forcing **(Yoshinobu)**

Assume $V \models \text{PFA}\big(= \text{FA(proper)}\big)$. Suppose $\mathbb{P}$ is $\omega_1 + 1$ operationally closed, as witnessed by strategy σ for player II. Suppose $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper".

(Yoshinobu) there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that: $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ is proper, and whenever $G * H * K$ is generic over V for it, then in $V[G * H * K]$ there exists a play $\mathcal{P}$ of length ω_1 with all proper initial segments in V, such that Player II used the strategy σ, and the conditions played in $\mathcal{P}$ generate G.

- Assume $j : V \to N$ is such that $\text{crit}(j) = \omega_2^V$ and:
  - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in \text{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
  - In N, there is $G * H * K$ generic over V for $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$.
- Then $H_\theta^V[G * H * K]$, and hence N, have the ω_1-length play $\mathcal{P}$ described above.
- Then $j[\mathcal{P}] \in N$, and is a play of length $j(\omega_1) = \omega_1$ according to $j(\sigma)$, hence has a lower bound. Also $j[\mathcal{P}]$ generates $j[G]$, so $j[G]$ has a lower bound in $j(\mathbb{P})$.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.
- $\dot{\mathbb{R}}$:= the $\mathbb{P} * \dot{\mathbb{Q}}$-name that shoots a club through $V \cap [H^V_{\omega_2}]^\omega$. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$.
- $\dot{\mathbb{R}}$:= the $\mathbb{P} * \dot{\mathbb{Q}}$-name that shoots a club through $V \cap [H^V_{\omega_2}]^\omega$. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.
- Assume $j : V \to N$ with $j[H^V_\theta] \in N$ etc, and such that there exists $G * H * K \in N$ generic over V.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.
- $\dot{\mathbb{R}}$:= the $\mathbb{P} * \dot{\mathbb{Q}}$-name that shoots a club through $V \cap [H^V_{\omega_2}]^\omega$. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.
- Assume $j : V \to N$ with $j[H^V_\theta] \in N$ etc, and such that there exists $G * H * K \in N$ generic over V.
- $H^V_\theta[G * H * K] \models H^V_{\omega_2} \in \mathrm{IC}$.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2}))$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.
- $\dot{\mathbb{R}}:=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name that shoots a club through $V \cap [H^V_{\omega_2}]^\omega$. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.
- Assume $j : V \to N$ with $j[H^V_\theta] \in N$ etc, and such that there exists $G * H * K \in N$ generic over V.
- $H^V_\theta[G * H * K] \models H^V_{\omega_2} \in \mathrm{IC}$.
- This is upward absolute to N. So N believes $j[G]^\frown H^V_{\omega_2}$ is a condition in $j(\mathbb{P})$.

Example: PFA does not separate IC from IS (or even from IU) **(C.)**

Assume PFA. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}(\mathrm{IC} \cap P_{\omega_2}(H_{\omega_2}))$ that are continuous at ω_1-cofinal points. Then $V^{\mathbb{P}} \models \mathrm{IC} =^* \mathrm{IU}$. Need to show PFA preserved.

- Assume $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". In particular, $V \cap [H^V_{\omega_2}]^\omega$ is stationary in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$.
- $\dot{\mathbb{R}}$:= the $\mathbb{P} * \dot{\mathbb{Q}}$-name that shoots a club through $V \cap [H^V_{\omega_2}]^\omega$. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.
- Assume $j : V \to N$ with $j[H^V_\theta] \in N$ etc, and such that there exists $G * H * K \in N$ generic over V.
- $H^V_\theta[G * H * K] \models H^V_{\omega_2} \in \mathrm{IC}$.
- This is upward absolute to N. So N believes $j[G]^\frown H^V_{\omega_2}$ is a condition in $j(\mathbb{P})$.

By the Preservation Theorem, PFA holds in $V^{\mathbb{P}}$.

Preserving “plus” versions of forcing axioms

Assume Γ is “nice” class of posets.

Preserving "plus" versions of forcing axioms

Assume Γ is "nice" class of posets.

Theorem (Forcing Axiom Preservation; see Thm 20 of Cox [1])

Assume FA (Γ), and $\mathbb{P}$ has the property: if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$, then there exists a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that:

- $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in \Gamma$;
- **Whenever** *$j : V \to N$ is a generic elementary embedding with $crit(j) = \omega_2^V$ and:*
  - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in wfp(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
  - *in N, there exists a $(V, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$-generic object $G * H * K$;*

  then *N believes that $j[G]$ has a lower bound in $j(\mathbb{P})$.*

Then $V^{\mathbb{P}} \models FA \quad (\Gamma)$.

Preserving "plus" versions of forcing axioms

Assume Γ is "nice" class of posets.

Theorem (Forcing Axiom Preservation; see Thm 20 of Cox [1])

Assume $FA^{++}(\Gamma)$, and $\mathbb{P}$ has the property: if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$, then there exists a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ such that:

- $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in \Gamma$;
- $\mathbb{P} * \dot{\mathbb{Q}}$ *forces "$\dot{\mathbb{R}}_{\dot{\mathbb{Q}}}$ preserves stationary subsets of ω_1";*
- **Whenever** $j : V \to N$ *is a generic elementary embedding with $crit(j) = \omega_2^V$ and:*
  - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}} \in H_\theta^V \in wfp(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
  - *in N, there exists a $(V, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_{\dot{\mathbb{Q}}})$-generic object $G * H * K$;*
  - $N \models$ *"$H_\theta^V[G * H * K]$ is correct about stationarity of subsets of ω_1";*

  then *N believes that $j[G]$ has a lower bound in $j(\mathbb{P})$.*

Then $V^{\mathbb{P}} \models FA^{++}(\Gamma)$.

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathsf{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathsf{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathsf{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**
3. Then $\mathbb{P} * \dot{\mathbb{Q}} (* \dot{\mathbb{R}}) \in \Gamma$ by "niceness" of Γ.

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathsf{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
   - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathsf{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
    - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
    - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}})$ and such that $N \models$ "$H_\theta^V[G * H(*K)]$ is correct about stationary subsets of ω_1".

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathrm{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
    - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
    - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}})$ and such that $N \models$ "$H_\theta^V[G * H(*K)]$ is correct about stationary subsets of ω_1".
5. Then $j[G] \in N$, and has size ω_1 there. By directed closure of $j(\mathbb{P})$ in N, $j[G]$ has a lower bound.

Example: preservation of "plus" version of FA under $< \omega_2$ directed closed (Larson)

Assume $\mathrm{FA}^{++}(\Gamma)$, and $\mathbb{P}$ is $< \omega_2$-directed closed.

1. Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{\Gamma}$.
2. Let $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for the trivial poset. **This is trivially ω_1-SSP**
3. Then $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}) \in \Gamma$ by "niceness" of Γ.
4. Assume $j : V \to N$ is such that $\mathrm{crit}(j) = \omega_2^V$ and:
   - $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \in H_\theta^V \in \mathrm{wfp}(N)$, $|H_\theta^V|^N = \omega_1$, $j \restriction H_\theta^V \in N$;
   - In N, there is $G * H(*K)$ that is generic over V for $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}})$ and such that $N \models$ "$H_\theta^V[G * H(*K)]$ is correct about stationary subsets of ω_1".
5. Then $j[G] \in N$, and has size ω_1 there. By directed closure of $j(\mathbb{P})$ in N, $j[G]$ has a lower bound.

So by the "plus" version of Preservation Theorem, $\mathrm{FA}^{++}(\Gamma)$ holds in $V^{\mathbb{P}}$.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H_\theta^V[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through IS $\cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}(\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2}))$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H_\theta^V[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

- By properness, $H_{\omega_2}^V \cap [H_{\omega_2}^V]^\omega$ is stationary in $V[G * H * K]$.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H_\theta^V[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

- By properness, $H_{\omega_2}^V \cap [H_{\omega_2}^V]^\omega$ is stationary in $V[G * H * K]$.
- w.l.o.g. H collapsed ω_2. So $H_\theta^V[G * H * K] \models H_{\omega_2}^V \in \mathsf{IS}$.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathsf{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H^V_\theta[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

- By properness, $H^V_{\omega_2} \cap [H^V_{\omega_2}]^\omega$ is stationary in $V[G * H * K]$.
- w.l.o.g. H collapsed ω_2. So $H^V_\theta[G * H * K] \models H^V_{\omega_2} \in \mathsf{IS}$.
- By (1), $N \models H^V_{\omega_2} \in \mathsf{IS}$ too.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H_\theta^V[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

- By properness, $H_{\omega_2}^V \cap [H_{\omega_2}^V]^\omega$ is stationary in $V[G * H * K]$.
- w.l.o.g. H collapsed ω_2. So $H_\theta^V[G * H * K] \models H_{\omega_2}^V \in \mathrm{IS}$.
- By (1), $N \models H_{\omega_2}^V \in \mathrm{IS}$ too.
- Then $N \models j[G]^\frown H_{\omega_2}^V$ is a condition.

Example: PFA^{++} does not separate IS from IU **(C.)**

Assume PFA^{++}. Let $\mathbb{P}$ shoot an ω_1-club through $\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})$. Conditions: $\subseteq$-increasing chains in ${}^{<\omega_2}\big(\mathrm{IS} \cap P_{\omega_2}(H_{\omega_2})\big)$ that are continuous at ω_1-cofinal points. Order by end-extension.

- Assume $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper. Let $\dot{\mathbb{R}}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset. Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and 3rd step is ω_1-SSP.
- Assume $j : V \to N$ as usual, and there is $G * H * K \in N$ generic over V such that

$$N \models H_\theta^V[G * H * K] \text{ is correct about stationary subsets of } \omega_1. \tag{1}$$

- By properness, $H_{\omega_2}^V \cap [H_{\omega_2}^V]^\omega$ is stationary in $V[G * H * K]$.
- w.l.o.g. H collapsed ω_2. So $H_\theta^V[G * H * K] \models H_{\omega_2}^V \in \mathrm{IS}$.
- By (1), $N \models H_{\omega_2}^V \in \mathrm{IS}$ too.
- Then $N \models j[G]^\frown H_{\omega_2}^V$ is a condition.

By the Preservation Theorem, PFA^{++} holds in $V^{\mathbb{P}}$.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}(“proper posets that force $H_{\omega_2}^V \in \mathsf{IS} \setminus \mathsf{IC}$”) **(C.)**

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

First, why consider this weird forcing axiom? Recall:

Theorem (C.)

DRP_{IS} does not imply RP_{IC}.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$”) **(C.)**

First, why consider this weird forcing axiom? Recall:

Theorem (C.)

DRP_{IS} does not imply RP_{IC}.

Proof has 2 parts:

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

First, why consider this weird forcing axiom? Recall:

Theorem (C.)

DRP_{IS} does not imply RP_{IC}.

Proof has 2 parts:

1. FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") implies $\mathsf{DRP}_{\mathsf{IS}}$.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

First, why consider this weird forcing axiom? Recall:

Theorem (C.)

DRP_{IS} does not imply RP_{IC}.

Proof has 2 parts:

1. FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") implies $\mathsf{DRP}_{\mathsf{IS}}$.
   - (fairly straightforward: apply the forcing axiom using, say, the poset $\mathrm{Add}(\omega) * \dot{\mathrm{Col}}(\omega_1, H_{(2^\theta)^+})$, and argue that you get "internal diagonal reflection").

Example: killing $\mathrm{RP}_{\mathrm{IC}}$ preserves FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$”) **(C.)**

First, why consider this weird forcing axiom? Recall:

Theorem (C.)

DRP_{IS} does not imply RP_{IC}.

Proof has 2 parts:

1. FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathrm{IS} \setminus \mathrm{IC}$”) implies $\mathrm{DRP}_{\mathrm{IS}}$.
   - (fairly straightforward: apply the forcing axiom using, say, the poset $\mathrm{Add}(\omega) * \dot{\mathrm{Col}}(\omega_1, H_{(2^\theta)^+})$, and argue that you get “internal diagonal reflection”).
2. That forcing axiom is preserved by killing $\mathrm{RP}_{\mathrm{IC}}$ (**next slide**)

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition".

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition". (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$”) **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and “no new countable set is covered by union of old condition”. (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition". (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.
- $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset, which is trivially ω_1-SSP.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}(“proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$”) **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and “no new countable set is covered by union of old condition”. (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.
- $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset, which is trivially ω_1-SSP.
- Assume $j : V \to N$ as in theorem, and that there is $G * H * K \in N$ generic over V s.t. $N \models$ “$H^V_\theta[G * H * K]$ is correct about stat subsets of ω_1”.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition". (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.
- $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset, which is trivially ω_1-SSP.
- Assume $j : V \to N$ as in theorem, and that there is $G * H * K \in N$ generic over V s.t. $N \models$ "$H^V_\theta[G * H * K]$ is correct about stat subsets of ω_1".
- By assumptions about $\dot{\mathbb{Q}}$, $V[G * H] = V[G * H * K] \models H^V_{\omega_2} = H^{V[G]}_{\omega_2} \notin \mathsf{IC}$.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition". (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.
- $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset, which is trivially ω_1-SSP.
- Assume $j : V \to N$ as in theorem, and that there is $G * H * K \in N$ generic over V s.t. $N \models$ "$H^V_\theta[G * H * K]$ is correct about stat subsets of ω_1".
- By assumptions about $\dot{\mathbb{Q}}$, $V[G * H] = V[G * H * K] \models H^V_{\omega_2} = H^{V[G]}_{\omega_2} \notin \mathsf{IC}$.
- By stat correctness of $V[G * H * K]$ in N, $N \models H^V_{\omega_2} = H^{V[G]}_{\omega_2} \notin \mathsf{IC}$.

Example: killing $\mathsf{RP}_{\mathsf{IC}}$ preserves FA^{++}("proper posets that force $H^V_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$") **(C.)**

$\mathbb{P}$: (variant of Aspero et al) conditions are ω_1-sized subsets of $[\omega_2]^\omega$ that don't reflect to any member of IC. Order: extension and "no new countable set is covered by union of old condition". (kills $\mathsf{RP}_{\mathsf{IC}}$; is $< \omega_2$ distributive)

- Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset that forces $H^{V^{\mathbb{P}}}_{\omega_2} \in \mathsf{IS} \setminus \mathsf{IC}$. By $< \omega_2$ distributivity of $\mathbb{P}$, $H^V_{\omega_2} = H^{V^{\mathbb{P}}}_{\omega_2}$.
- $\dot{\mathbb{R}} :=$ the $\mathbb{P} * \dot{\mathbb{Q}}$-name for trivial poset, which is trivially ω_1-SSP.
- Assume $j : V \to N$ as in theorem, and that there is $G * H * K \in N$ generic over V s.t. $N \models$ "$H^V_\theta[G * H * K]$ is correct about stat subsets of ω_1".
- By assumptions about $\dot{\mathbb{Q}}$, $V[G * H] = V[G * H * K] \models H^V_{\omega_2} = H^{V[G]}_{\omega_2} \notin \mathsf{IC}$.
- By stat correctness of $V[G * H * K]$ in N, $N \models H^V_{\omega_2} = H^{V[G]}_{\omega_2} \notin \mathsf{IC}$. Then $\cup G = j[\cup G]$ is a condition in $j(\mathbb{P})$.

Example: killing WRP(ω_2) preserves FA^{++}(“SSP posets that force $H^V_{\omega_2} \notin$ IS”) (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and “no new countable set is covered by union of old condition”. Is $< \omega_2$-distributive, and kills WRP).

Example: killing WRP(ω_2) preserves FA^{++}("SSP posets that force $H_{\omega_2}^V \notin$ IS") (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and "no new countable set is covered by union of old condition". Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H_{\omega_2}^{V^{\mathbb{P}}} = H_{\omega_2}^V \notin$ IS.

Example: killing WRP(ω_2) preserves FA^{++}(“SSP posets that force $H^V_{\omega_2} \notin$ IS”) (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and “no new countable set is covered by union of old condition”. Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H^{V^{\mathbb{P}}}_{\omega_2} = H^V_{\omega_2} \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.

Example: killing WRP(ω_2) preserves FA^{++}(“SSP posets that force $H^V_{\omega_2} \notin$ IS”) (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and “no new countable set is covered by union of old condition”. Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H^{V^{\mathbb{P}}}_{\omega_2} = H^V_{\omega_2} \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.
- Assume $j : V \to N$ as usual, and $G * H * K \in N$ generic over V s.t. $N \models$ “$V[G * H * K]$ is correct about stationary subsets of ω_1”.

Example: killing WRP(ω_2) preserves FA^{++}("SSP posets that force $H^V_{\omega_2} \notin$ IS") (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and "no new countable set is covered by union of old condition". Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H^{V^{\mathbb{P}}}_{\omega_2} = H^V_{\omega_2} \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.
- Assume $j : V \to N$ as usual, and $G * H * K \in N$ generic over V s.t. $N \models$ "$V[G * H * K]$ is correct about stationary subsets of ω_1".
- $\cup G \subset V \cap [\omega_2]^\omega$, and the latter is nonstationary in $V[G * H]$ (by assumption about $\dot{\mathbb{Q}}$), and hence in N.

Example: killing WRP(ω_2) preserves FA^{++}("SSP posets that force $H_{\omega_2}^V \notin$ IS") (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and "no new countable set is covered by union of old condition". Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H_{\omega_2}^{V^{\mathbb{P}}} = H_{\omega_2}^V \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.
- Assume $j : V \to N$ as usual, and $G * H * K \in N$ generic over V s.t. $N \models$ "$V[G * H * K]$ is correct about stationary subsets of ω_1".
- $\cup G \subset V \cap [\omega_2]^\omega$, and the latter is nonstationary in $V[G * H]$ (by assumption about $\dot{\mathbb{Q}}$), and hence in N.
- Hence $N \models \cup G$ is a condition in $j(\mathbb{P})$.

Example: killing WRP(ω_2) preserves FA^{++}("SSP posets that force $H^V_{\omega_2} \notin$ IS") (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and "no new countable set is covered by union of old condition". Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H^{V^{\mathbb{P}}}_{\omega_2} = H^V_{\omega_2} \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.
- Assume $j : V \to N$ as usual, and $G * H * K \in N$ generic over V s.t. $N \models$ "$V[G * H * K]$ is correct about stationary subsets of ω_1".
- $\cup G \subset V \cap [\omega_2]^\omega$, and the latter is nonstationary in $V[G * H]$ (by assumption about $\dot{\mathbb{Q}}$), and hence in N.
- Hence $N \models \cup G$ is a condition in $j(\mathbb{P})$. (and extends $G = j[G]$)

Example: killing WRP(ω_2) preserves FA^{++}(“SSP posets that force $H_{\omega_2}^V \notin$ IS”) (**C.**)

$\mathbb{P}$: Aspero et al. poset to add nonreflecting subset of $[\omega_2]^\omega$ (subsets of $[\omega_2]^\omega$ of size ω_1 that don't reflect; order by extension and “no new countable set is covered by union of old condition”. Is $< \omega_2$-distributive, and kills WRP).

- Let $\dot{\mathbb{Q}}$ be $\mathbb{P}$-name for SSP poset forcing $H_{\omega_2}^{V^{\mathbb{P}}} = H_{\omega_2}^V \notin$ IS.
- $\dot{\mathbb{R}} := \mathbb{P} * \dot{\mathbb{Q}}$ name for trivial poset.
- Assume $j : V \to N$ as usual, and $G * H * K \in N$ generic over V s.t. $N \models$ “$V[G * H * K]$ is correct about stationary subsets of ω_1”.
- $\cup G \subset V \cap [\omega_2]^\omega$, and the latter is nonstationary in $V[G * H]$ (by assumption about $\dot{\mathbb{Q}}$), and hence in N.
- Hence $N \models \cup G$ is a condition in $j(\mathbb{P})$. (and extends $G = j[G]$)

By Preservation Theorem, the FA holds in $V^{\mathbb{P}}$.

Outline

Summary

General theorem about forcing axiom preservation:

- provided new results, and considerable simplification (and strengthening) of some prior results related to internal approachability;

Summary

General theorem about forcing axiom preservation:

- provided new results, and considerable simplification (and strengthening) of some prior results related to internal approachability;
- provides a uniform framework for many (all?) FA preservation results in the literature;

Summary

General theorem about forcing axiom preservation:

- provided new results, and considerable simplification (and strengthening) of some prior results related to internal approachability;
- provides a uniform framework for many (all?) FA preservation results in the literature;
- But also was used to show that similar attempts to settle Krueger, Beaudoin questions must fail.

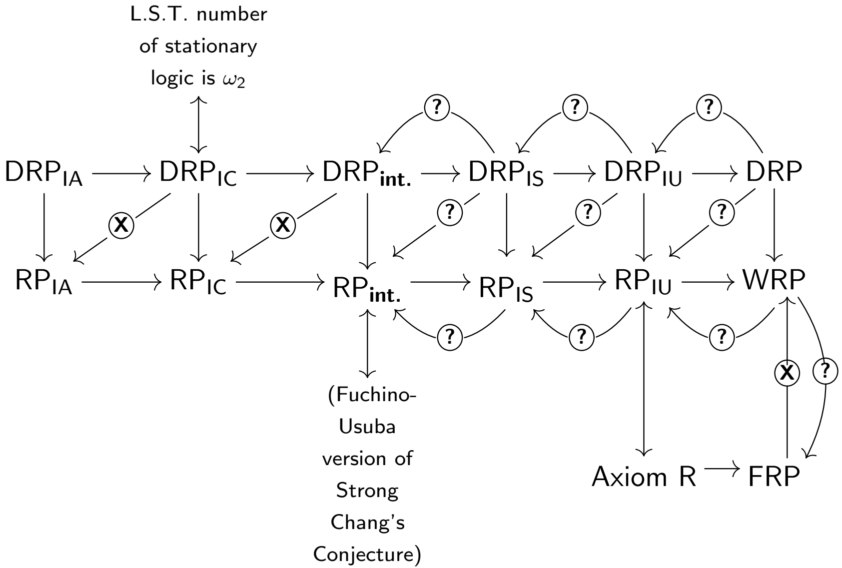
Summary

General theorem about forcing axiom preservation:

- provided new results, and considerable simplification (and strengthening) of some prior results related to internal approachability;
- provides a uniform framework for many (all?) FA preservation results in the literature;
- But also was used to show that similar attempts to settle Krueger, Beaudoin questions must fail.

There is a similar "lifting embeddings" theorem for preservation of fragments of forcing axioms; e.g. where $V \models \mathsf{FA}(\Gamma)$ and you want $V^{\mathbb{P}}$ to model $\mathsf{FA}(\dot{\Psi})$ for some other (narrower) class $\dot{\Psi}$.

Current status: RHS is wide open



[1] Sean Cox, *Forcing axioms, approachability at ω_2, and stationary set reflection*, under review, available at https://arxiv.org/abs/1807.06129.