On Countable Stationary Towers

Yo Matsubara
Nagoya University

Joint work with
Toshimichi Usuba (Waseda University)

Theorem (H. Woodin)

If there exists a supercompact cardinal, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable and has the Baire property and the perfect set property.

Abbreviation : We denote the conclusion of this theorem by “$L(\mathbb{R})$ satisifies RP (the regularity properties of reals)”.

Woodin's Lemma

Let λ be an inaccessible cardinal.
Suppose there is a p.o. $\mathbb{P}$ such that $\mathbb{P}$ forces the following:
There is a generic elementary embedding $j : V \to M$ with

- (1) $crit(j) = (\omega_1)^V$ $\quad j(\omega_1) = \lambda$
- (2) ${}^{\omega}M \subseteq M$ (i.e M is closed under taking ω sequences in $V^{\mathbb{P}}$)
- (3) every real is obtained by a small (size less than λ) forcing

Then $L(\mathbb{R})$ satisifies RP.

By "every real is obtained by a small forcing", we mean that in $V[G]$ where G is $\mathbb{P}$ -generic, for each real x there are a poset $\mathbb{P}_x$ in the ground model V with cardinality less than λ and a $\mathbb{P}_x$ -generic $H \in V[G]$ such that $x \in V[H]$.

With this property, one can show that the reals in M $(= \mathbb{R}^{V[G]})$ are the reals in a generic extension of V by Levy collapsing our inaccessible λ. Hence, by Solovay's theorem, M thinks that $L(\mathbb{R})$ satisfies RP. So, by the elementarity of j, $L(\mathbb{R})$ satisfies RP in V.

Countable Stationary Towers

<u>Definition</u> Let λ be a limit ordinal $>\omega_1$.
Let $\mathbb{Q}_\lambda$ denote the set

$$\{p \in V_\lambda : p \;\; is \;\; stationary \; in \; \wp_{\omega_1}(\bigcup p)\} \quad .$$

For $p, q \in \mathbb{Q}_\lambda$, we let
$p \leq_{\mathbb{Q}_\lambda} q$ if "$\cup\, q \subseteq \cup\, p \quad \wedge \forall x \in p(x \cap (\cup\, q) \;\; \in q)$"

- The p.o. $\langle \mathbb{Q}_\lambda, \leq_{\mathbb{Q}_\lambda} \rangle$ is known as the *countable stationary tower of height* λ.

Remark

If λ is an inaccessible cardinal, then $\mathbb{Q}_\lambda$ is forcing equivalent to the poset associated to the tower $\langle NS_{\omega_1\alpha}\colon\ \alpha < \lambda\rangle$ where $NS_{\omega_1\alpha}$ is the nonstationary ideal over $\wp_{\omega_1}\alpha$.

If G is a $\mathbb{Q}_\lambda$-generic filter over V, just as with the nonstationary ideal, one can take the generic ultrapower $Ult(V;G)$ of V.

In general, there is no guarantee that $Ult(V;G)$ is well-founded.

<u>Definition</u>

(i) $\mathbb{Q}_\lambda$ is *precipitous* if $\mathbb{Q}_\lambda$ forces "$Ult(V;G)$ is well-founded"

(ii) $\mathbb{Q}_\lambda$ is *presaturated* if $\mathbb{Q}_\lambda$ forces "$cof(\lambda) > \omega$"

If $|\mathbb{Q}_\lambda| = \lambda$, then presaturation of $\mathbb{Q}_\lambda$ implies that "${}^{\omega}Ult(V;G) \subseteq Ult(V;G)$"

i.e. $Ult(V;G)$ is closed under taking ω sequences in $V^{\mathbb{Q}_\lambda}$.

If $Ult(V;G)$ is well-founded, then, by taking its transitive collapse, we obtain a generic elementary embedding $j : V \to M \cong Ult(V;G)$ with $crit(j) = (\omega_1)^V$.

<u>Remark</u> If $\mathbb{Q}_\lambda$ is presaturated with $|\mathbb{Q}_\lambda| = \lambda$, then it is precipitous. Furthermore if $j : V \to M$ is a generic elementary embedding associated with such $\mathbb{Q}_\lambda$, then $j(\omega_1) = \lambda$ and ${}^\omega M \subseteq M$ in $V^{\mathbb{Q}_\lambda}$.

Theorem (Woodin)

If λ is a Woodin cardinal, then $\mathbb{Q}_\lambda$ is presaturated.

Remark   The existence of one Woodin cardinal is not enough to imply " $L(\mathbb{R})$ satisfies RP" .
Even ω many Woodin cardinals cannot imply $L(\mathbb{R})$ satisfies RP .
Woodin showed the consistency of "ω many Woodin cardinals + $L(\mathbb{R})$ satisfies AC "
from the consistency of ZFC +ω many Woodin cardinals.

If we let δ denote $|\mathbb{Q}_\lambda|^+$, then the well-foundedness of $Ult(V_\delta\,; G)$ is equivalent to the well-foundedness of $Ult(V; G)$.

So for an inaccessible λ, the precipitousness of $\mathbb{Q}_\lambda$ is equivalent to "$\mathbb{Q}_\lambda$ forces $Ult(V_{\lambda^+};\ G)$ is well-founded".

This motivated the next definition.

<u>Definition</u>

$\mathbb{Q}_\lambda$ is *semi-precipitous* if $\mathbb{Q}_\lambda$ forces "$Ult(V_\lambda; G)$ is well-founded"

We are interested in the following questions:

1. Can we prove "$L(\mathbb{R})$ satisfies RP" from the properties of $\mathbb{Q}_\lambda$'s such as precipitousness or presaturation without explicit use of strong large cardinals like Woodin cardinals ?

2. Can $\mathbb{Q}_\lambda$ be precipitous when λ is not a large cardinal? For example, can $\mathbb{Q}_\lambda$ be precipitous for a singular cardinal λ? We know that $\mathbb{Q}_\lambda$ cannot be presaturated if λ is a singular cardinal with $|\mathbb{Q}_\lambda| = \lambda$.

3. Is the semi-precipitousness of $\mathbb{Q}_\lambda$ a "large cardinal" property?

Results

Theorem1

Suppose $\mathbb{Q}_\alpha$ is precipitous.
Then for each $p \in \mathbb{Q}_\alpha$, there is a stationary subset p^* of $\mathcal{P}_{\omega_1} V_{\alpha+1}$ such that , for every limit ordinal $\beta > \alpha$, $p^* \leq_{\mathbb{Q}_\beta} p$ and $p^* \Vdash_{\mathbb{Q}_\beta}$ "$\dot{G} \cap \mathbb{Q}_\alpha$ is $\mathbb{Q}_\alpha$-generic" where $\dot{G}$ is the canonical name for a $\mathbb{Q}_\beta$-generic filter .

Note If $\exists \delta > \alpha$ ($\mathbb{Q}_\delta$ is semi-precipitous), then the conclusion of the above theorem implies the precipitousness of $\mathbb{Q}_\alpha$.

To prove Theorem1, we used the following game Γ_α:

Players I and II alternately play elements of $\mathbb{Q}_\alpha$ such that $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots$ where p_n's are I's moves and q_n's are II's moves. Player II wins this game iff there is some $x \in [V_\alpha]^\omega$ such that $x \cap \bigcup p_n \in p_n$ for every $n < \omega$.

<u>Theorem</u> (Doug Burke)
Let α be a limit ordinal $> \omega$. Then the following are equivalent:

(1) $\mathbb{Q}_\alpha$ is precipitous.

(2) Player I does not have a winning strategy in the game Γ_α .

We used the next claim to prove Theorem 1:

Claim If $\mathbb{Q}_\alpha$ is precipitous, then for every $p \in \mathbb{Q}_\alpha$, the following set p^* is stationary:

$$p^* = \{ x \in [V_{\alpha+1}]^\omega \colon x \cap \cup p \in p \wedge \forall D \in x(D\ is\ dense\ in\ \mathbb{Q}_\alpha \to \exists r \in D \cap x(x \cap \cup r \in r)\}$$

We can prove this claim by producing a winning strategy for Player I in the game Γ_α using the assumption of the above set being non-stationary.

Theorem 1, together with Woodin's results, imply the next result.

Theorem 2

If λ is an inaccessible cardinal such that
(i) $\mathbb{Q}_\lambda$ is presaturated and
(ii) the set $\{\alpha < \lambda : \mathbb{Q}_\alpha \ is \ precipitous\}$ is unbounded in λ,
then $L(\mathbb{R})$ satisfies RP.

Note: By Theorem 1, (ii) of Theorem 2 implies that $\mathbb{Q}_\lambda$ forces
$\{\alpha < \lambda: G \cap \mathbb{Q}_\alpha \ is \ \mathbb{Q}_\alpha \ generic\}$ to be unbounded in λ where G is $\mathbb{Q}_\lambda$-generic.

We obtained some results concerning the notion of semi-precipitousness.

<u>Theorem 3</u>

Suppose $|V_\lambda| = \lambda$ and $\omega < cof(\lambda) < \lambda$.
If $\mathbb{Q}_\lambda$ is semi-precipitous, then it is precipitous.

The precipitousness of $\mathbb{Q}_\lambda$ where λ is an inaccessible cardinal implies the existence of many α's $< \lambda$ where $\mathbb{Q}_\alpha$ is semi-precipitous.

Theorem 4

If $\mathbb{Q}_\lambda$ is precipitous where λ is inaccessible , then$\{\alpha < \lambda : cof(\alpha) = \omega \ \wedge \ \mathbb{Q}_\alpha \ is \ \ semiprecipitous\}$ is stationary in λ.

Furthermore, if λ is a Mahlo cardinal, then $\{\alpha < \lambda : \alpha \ is \ inaccessible \ \wedge \ \mathbb{Q}_\alpha \ is \ \ semiprecipitous\}$ is stationary in λ.

The last result indicates that "semi-precipitous" cannot imply "precipitousness" in general.

Otherwise, from λ being a Woodin cardinal, using Theorem 2 and Theorem 4, we get "$L(\mathbb{R})$ satisfies RP". But, by Woodin's work, one Woodin cardinal cannot imply that conclusion.

For an inaccessible λ, $\{\alpha < \lambda : \mathbb{Q}_\alpha \ is \ precipitous\}$ being stationary has interesting consequences.

<u>Theorem 5</u>

Suppose λ is an inaccessible cardinal.
If $\{\alpha < \lambda : \mathbb{Q}_\alpha \ is \ precipitous\}$ is stationary, then $\mathbb{Q}_\lambda$ is precipitous. In fact, it is presaturated,

<u>Theorem 6</u>

If $\mathbb{Q}_\lambda$ is precipitous where λ is a weakly compact cardinal, then$\{\alpha < \lambda : \mathbb{Q}_\alpha \ is \ precipitous\}$ is stationary, impling the presaturation of $\mathbb{Q}_\lambda$.

Theorem 7

If $\mathbb{Q}_{\lambda}$ is precipitous where λ is a weakly compact cardinal, then $L(\mathbb{R})$ satisfies RP.

Questions

1. Can we derive "$L(\mathbb{R})$ satisfies AD" from our hypotheses concerning the properties of countable stationary towers?

2. We still do not have answers to the following problems;

(i) Can $\mathbb{Q}_\lambda$ be precipitous where λ is not a large cardinal?

(ii) Is semi-precipitousness of $\mathbb{Q}_\lambda$ a "large cardinal" property in general?

We know the following;

"$\mathbb{Q}_\lambda$ is semi-precipitous + $\mathbb{Q}_\lambda$ forces $j(\omega_1) = \lambda$" implies $\exists 0^{\#}$.