The special $\aleph_2$-Aronszajn tree property and GCH

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Aronszajn trees and Suslin trees

Let κ be a regular uncountable cardinal.

- A κ-tree is a tree T of height κ all of whose levels are smaller than κ. A *κ-Aronszajn tree* is a κ-tree which has no κ-branches.
- A *κ-Suslin tree* is a κ-tree which has no κ-branches and no antichains of size κ.
- If $\kappa = \lambda^+$, a κ-Aronszajn tree T is said to be *special* if there exists a function $f : T \to \lambda$ such that $f(x) \neq f(y)$ whenever $x, y \in T$ are such that $x <_T y$.

Aronszajn trees were introduced by Kurepa, and Aronszajn (1934) proved the existence, in ZFC, of a special $\aleph_1$-Aronszajn tree. Later, Specker (1949) showed that $2^{<\lambda} = \lambda$ implies the existence of special λ^+-Aronszajn trees for λ regular, and Jensen (1972) produced special λ^+-Aronszajn trees for singular λ in L.

Baumgartner, Malitz and Reinhardt (1970) showed that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies that all $\aleph_1$-Aronszajn trees are special. In particular, under this assumption there are no $\aleph_1$-Suslin trees. Later, Jensen (1974) produced a model of GCH in which there are no $\aleph_1$-Suslin trees, and in fact all $\aleph_1$-Aronszajn trees are special.

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$\aleph_2$-Suslin trees

The situation at $\aleph_2$ turned out to be more complicated. Jensen (1972) proved that the existence of an $\aleph_2$-Suslin tree follows from each of the hypotheses CH $+\diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\})$ and $\Box_{\omega_1} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\})$. The second result was improved by Gregory (1976); he proved that GCH together the existence of a non–reflecting stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ yields the existence of an $\aleph_2$-Suslin tree.

Theorem

(Laver–Shelah, 1981) If there is a weakly compact cardinal κ, then there is a forcing extension in which $\kappa = \aleph_2$, CH holds, and all $\aleph_2$-Aronszajn trees are special (and hence there are no $\aleph_2$-Suslin trees).

The proof proceeds by

- Lévy–collapsing κ to become ω_2, and then
- running a countable–support iteration of length κ^+ in which one specializes, with countable conditions, all κ-Aronszajn trees given by some book-keeping function.
- One uses the weak compactness of κ in V in a crucial way in order to show that the iteration has the κ-c.c. and hence everything goes as planned.

In the Laver–Shelah model, $2^{\aleph_1} = \aleph_3$, and the following remained a major open problem (s. e.g. Kanamori–Magidor 1977):

Question

Is ZFC+GCH consistent with the non–existence of $\aleph_2$-Suslin trees?

Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

Given a cardinal κ and $T \subseteq H(\kappa)$, a finite $\mathcal{N} \subseteq [H(\kappa)]^{\aleph_0}$ is a *T–symmetric system* if

(1) for every $N \in \mathcal{N}$,

$$(N, \in, T) \preccurlyeq (H(\kappa), \in, T),$$

(2) given N_0, $N_1 \in \mathcal{N}$, if $N_0 \cap \omega_1 = N_1 \cap \omega_1$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0, \in, T) \longrightarrow (N_1, \in, T)$$

and Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) Given N_0, $N_1 \in \mathcal{N}$ such that $N_0 \cap \omega_1 = N_1 \cap \omega_1$ and $M \in N_0 \cap \mathcal{N}$, $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

(4) Given M, $N_0 \in \mathcal{N}$ such that $M \cap \omega_1 < N_0 \cap \omega_1$, there is some $N_1 \in \mathcal{N}$ such that $N_1 \cap \omega_1 = N_0 \cap \omega_1$ and $M \in N_1$.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then $\Psi_{N,N'}$ is an isomorphism

$$\Psi_{N,N'} : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

Proof: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. **By symmetry,** $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$.
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In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

Something one may want to try at this point: Extend the symmetry requirements **also** to the working parts in such a way that the above CH–preservation argument goes trough. Hope to be able to force something interesting this way.

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While Visiting Mohammad Golshani in Tehran in December 2017, we thought about implementing these ideas (with $2^{\aleph_1} = \aleph_2$ instead of $2^{\aleph_0} = \aleph_1$ and $\aleph_1$-sized models instead if countable models) for the Laver–Shelah construction, in order to build a model of GCH with no $\aleph_2$-Suslin trees. We eventually succeeded:

The result

Theorem (A.–Golshani) Suppose κ is a weakly compact cardinal. Then there exists a set–generic extension of the universe in which

(1) GCH holds,

(2) $\kappa = \aleph_2$, and

(3) All $\aleph_2$-Aronszajn trees are special (and hence there are no $\aleph_2$-Suslin trees).

Proof sketch

Let κ be weakly compact. W.l.o.g. we may assume $2^\mu = \mu^+$ for all $\mu \geq \kappa$.
Let

$$\Phi : \kappa^+ \to H(\kappa^+)$$

be such that for each $x \in H(\kappa^+)$, $\Phi^{-1}(x)$ is an unbounded subset of κ^+. Φ exists by $2^\kappa = \kappa^+$.
Let also $(\Phi_\alpha)_{\alpha<\kappa^+}$ be the following sequence of subsets of $H(\kappa^+)$.

- $\Phi_0 = \Phi$
- If $\alpha > 0$, then Φ_α codes, in some fixed canonical way, the satisfaction predicate for the structure

  $$\langle H(\kappa^+), \in, \vec{\Phi}_\alpha \rangle,$$

  where $\vec{\Phi}_\alpha = (\Phi_{\alpha'})_{\alpha'<\alpha}$.

Let $\mathcal{F}$ be the weak compactness filter on κ, i.e., the filter on κ generated by the sets

$$\{\lambda < \kappa \mid (V_\lambda, \in, B \cap V_\lambda) \models \psi\},$$

where $B \subseteq V_\kappa$ and where ψ is a Π^1_1 sentence for the structure $(V_\kappa, \in, B)$ such that

$$(V_\kappa, \in, B) \models \psi$$

$\mathcal{F}$ is a proper normal filter on κ. Let also $\mathcal{S}$ be the collection of $\mathcal{F}$-positive subsets of κ, i.e.,

$$\mathcal{S} = \{X \subseteq \kappa \mid X \cap C \neq \emptyset \text{ for all } C \in \mathcal{F}\}$$

The main ingredient: Revisionism (copying information from the future into the past).

Let us call

$$\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$$

an *edge below* β if

(0) For all $i \in \{0, 1\}$, $N_i \subseteq H(\kappa^+)$, $\delta_{N_i} := N \cap \kappa \in \kappa$, $|N_i| = |\delta_{N_i}|$, and ${}^{<|N_i|}N_i \subseteq N_i$.

(1) For all $i \in \{0, 1\}$, γ_i is an ordinal in the closure of $N_i \cap \{\xi + 1 \ : \ \xi < \beta\}$ and $(N_i, \in, \Phi_\alpha) \preccurlyeq (H(\kappa^+), \in, \Phi_\alpha)$ for all $\alpha \in N_i \cap \gamma_i$.

(2) $N_0 \cong N_1$ via an isomorphism $\Psi_{N_0,N_1} : N_0 \longrightarrow N_1$ such that
   - (i) $(N_0, \in, \Phi_\alpha) \cong (N_1, \in, \Phi_{\Psi_{N_0,N_1}(\alpha)})$ for all $\alpha < \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$,
   - (ii) Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$, and
   - (iii) $\Psi_{N_0,N_1}(\xi) \leq \xi$ for every ordinal $\xi \in N_0$.

Given $\beta \leq \kappa^+$, we will build $\mathbb{Q}_\beta$ as a forcing with side conditions consisting of sets of edges below β.

Given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in the side condition, we will copy information in N_0 attached to $\alpha < \gamma_0$ via Ψ_{N_0,N_1} into N_1 if $\Psi_{N_0,N_1}(\alpha) < \gamma_1$.

We do not require that information in N_1 attached to $\Psi_{N_0,N_1}(\alpha)$ be copied into N_0.

Given models with markers (N, γ), (N_0, γ_0) and (N_1, γ_1), if $(N, \gamma) \in N_0$ and $(N_0, \in) \cong (N_1, \in)$, then we let $\pi^{\gamma,N}_{N_0,\gamma_0,N_1,\gamma_1}$ denote the supremum of the set of ordinals $\xi + 1 \in N_1$ such that

- $\xi < \Psi_{N_0,N_1}(\gamma)$,
- $\Psi_{N_1,N_0}(\xi) < \gamma_0$, and
- $\xi < \gamma_1$

Assuming $\alpha < \kappa^+$, $\mathbb{Q}_\alpha$ defined and

$$\Vdash_{\mathbb{Q}_\alpha} \mathsf{CH} + \kappa = \omega_2,$$

we let $\underset{\sim}{T}_\alpha \in H(\kappa^+)$ be a canonically chosen $\mathbb{Q}_\alpha$-name for κ-Aronszajn tree such that $\underset{\sim}{T}_\alpha = \Phi_\alpha$ if $\Phi(\alpha)$ is a $\mathbb{Q}_\alpha$-name for a κ-Aronszajn tree.

We assume that for each $\rho < \kappa$, the ρ-th level of $\underset{\sim}{T}_\alpha$ is $\{\rho\} \times \omega_1$.

Definition of the forcing

Let $\beta \leq \kappa^+$ and suppose $\mathbb{Q}_\alpha$ defined for all $\alpha < \beta$. A condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) f_q is a countable function such that $\mathrm{dom}(f_q) \subseteq \kappa^+ \cap \beta$ and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.

  (a) If $\alpha = 0$, then $f_q(\alpha) \in \mathrm{Col}(\omega_1, {<}\kappa)$.
  (b) If $\alpha > 0$, then

$$f_q(\alpha) : \kappa \times \omega_1 \to \omega_1$$

  is a countable function.

(2) τ_q is a countable set of edges below β.

(3) The following holds for every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$.

(a) If $\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in N_0 \cap \tau_q$, then

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*) \rangle \in \tau_q$$

for some $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0,\gamma_0,N_1,\gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0,\gamma_0,N_1,\gamma_1}$.

(b) The following holds for each nonzero ordinal $\alpha \in \mathrm{dom}(f_q) \cap N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$.

(i) $\Psi_{N_0,N_1}(\alpha) \in \mathrm{dom}(f_q)$

(ii) $f_q(\alpha) \restriction \delta_{N_0} \times \omega_1 \subseteq f_q(\Psi_{N_0,N_1}(\alpha))$

(4) For all $\alpha < \beta$, $q \restriction \alpha \in \mathbb{Q}_\alpha$, where

$$q \restriction \alpha = (f_q \restriction \alpha, \tau_q \restriction \alpha)$$

(5) The following holds for every nonzero $\alpha < \beta$.

(a) If $\alpha \in \mathrm{dom}(f_q)$, then $q \restriction \alpha$ forces that $f_q(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$.

(b) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$, if $\alpha \in N_0 \cap \gamma_0$, then $\mathbb{Q}_{\alpha+1} \cap N_0 \lessdot \mathbb{Q}^{N_0}_{\alpha+1}$, where $\mathbb{Q}^{N_0}_{\alpha+1}$ is the partial order whose conditions are ordered pairs $p = (f_p, \tau_p)$ such that

(i) f_p is a function such that $\mathrm{dom}(f_p) \subseteq \alpha + 1$,

(ii) if $\alpha \in \mathrm{dom}(f_p)$, then $f_p(\alpha) : \kappa \times \omega_1 \to \omega_1$ is a countable function,

(iii) τ_p is a set of edges below $\alpha + 1$,

(iv) $\gamma_0, \gamma_1 \leq \alpha$ for every $\langle (N_0', \gamma_0), (N_1', \gamma_1) \rangle \in \tau_p \setminus N_0$,

(v) $p \restriction \alpha \in \mathbb{Q}_\alpha$,

(vi) $p \restriction N_0 \in \mathbb{Q}_{\alpha+1}$, and

(vii) if $\alpha \in \mathrm{dom}(f_p)$, then $p \restriction \alpha$ forces that $f_p(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$,

ordered by setting $p_1 \leq_{\mathbb{Q}^{N_0}_{\alpha+1}} p_0$ if

- $p_1 \restriction \alpha \leq_{\mathbb{Q}_\alpha} p_0 \restriction \alpha$ and
- $f_{p_0}(\alpha) \subseteq f_{p_1}(\alpha)$ in case $\alpha \in \mathrm{dom}(f_{p_0})$.

The extension relation:
Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_\beta q_0$ (q_1 is an extension of q_0) if and only if the following holds.

(A) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$

(B) for every $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$.

(C) For every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q_0}$ there are $\gamma'_0 \geq \gamma_0$ and $\gamma'_1 \geq \gamma_1$ such that $\langle (N_0, \gamma'_0), (N_1, \gamma'_1) \rangle \in \tau_{q_1}$.

Main facts

(0) For every $\beta < \kappa^+$, $\mathbb{Q}_\beta$ is definable over the structure

$$(H(\kappa^+), \in, \Phi_{\beta+1})$$

without parameters. Moreover, this definition can be taken to be uniform in β.

(1) $\mathbb{Q}_1$ forces $\kappa = \omega_2$.

(2) For every $\beta \leq \kappa^+$,

 (i) $\mathbb{Q}_\alpha \subseteq \mathbb{Q}_\beta$ for all $\alpha < \beta$, and
 (ii) if $\mathrm{cf}(\beta) \geq \kappa$, then $\mathbb{Q}_\beta = \bigcup_{\alpha<\beta} \mathbb{Q}_\alpha$.

(3) Thanks to the fact that we are only copying information 'from the future into the past', $(\mathbb{Q}_\beta)_{\beta\leq\kappa^+}$ is a forcing iteration (i.e., $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_\beta$ for all $\alpha < \beta$): Given $q \in \mathbb{Q}_\beta$ and $r \in \mathbb{Q}_\alpha$, if $r \leq_\alpha q \restriction \alpha$, then

$$(f_r \cup f_q \restriction [\alpha, \beta), \tau_q \cup \tau_r)$$

is a common extension of q and r in $\mathbb{Q}_\beta$.

(4) $\mathbb{Q}_\beta$ is σ-closed for every $\beta \leq \kappa^+$. In fact, every decreasing ω-sequence $(f_n)_{n<\omega}$ of $\mathbb{Q}_\beta$-conditions has a greatest lower bound q^* in $\mathbb{Q}_\beta$, $q^* = (f, \bigcup_n \tau_{q_n})$, where $\mathrm{dom}(f) = \bigcup_n \mathrm{dom}(f_{q_n})$, and

$$f(\alpha) = \bigcup\{f_{q_m}(\alpha) \,:\, m \geq n\}$$

for all n and $\alpha \in \mathrm{dom}(f_{q_n})$. In particular, forcing with $\mathbb{Q}_\beta$ does not add new ω-sequences of ordinals, and therefore it preserves both ω_1 and CH.

(5) $\mathbb{Q}_{\kappa^+}$ has the κ-c.c.

(6) $\mathbb{Q}_{\kappa^+}$ adds κ-many new subsets of ω_1, but not more than that; in particular, $\mathbb{Q}_{\kappa^+}$ preserves $2^{\aleph_1} = \aleph_2$ [essentially the same argument we saw a few slides back].

(7) $\mathbb{Q}_{\kappa^+}$ forces that all $\aleph_2$-Aronszajn are special.

Proof of (6): $\mathbb{Q}_{\kappa^+}$ adds less than κ^+-many new subsets of ω_1.

Suppose, towards a contradiction, that $q \in \mathbb{Q}_{\kappa^+}$ and there is a sequence $(\underset{\sim}{r}_i)_{i<\kappa^+}$ of names for subsets of ω_1 such that

$$q \Vdash_{\mathbb{Q}_{\kappa^+}} \underset{\sim}{r}_i \neq \underset{\sim}{r}_{i'} \text{ for all } i < i' < \kappa^+$$

By κ-c.c. we may assume, for each i, that $\underset{\sim}{r}_i \in H(\kappa^+)$ and $\underset{\sim}{r}_i$ is a $\mathbb{Q}_{\beta_i}$-name for some $\beta_i < \kappa^+$.

Let θ be a large enough. For each $i < \kappa^+$ let $N_i^* \preceq H(\theta)$ be such that

(1) $|N_i^*| = |N_i^* \cap \kappa|$,

(2) N_i^* is closed under sequences of length less than $|N_i^*|$,

(3) q, $\underset{\sim}{r}_i$, β_i, $(\Phi_\alpha)_{\alpha<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha<\kappa^+} \in N_i^*$, and

(4) $\mathbb{Q}_\alpha \cap N_i^* \lessdot \mathbb{Q}_\alpha$ for every $\alpha \in \kappa^+ \cap N_i^*$.

N_i^* can be found by a Π^1_1-reflection argument, using the weak compactness of κ and the κ-c.c. of all $\mathbb{Q}_\alpha$:

Let $M^* \preceq H(\theta)$ be such that

- q, $\underset{\sim}{r}_i$, β_i, $(\Phi_\alpha)_{\alpha<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha<\kappa^+} \in M^*$,
- $|M^*| = \kappa$, and
- ${}^{<\kappa}M^* \subseteq M^*$.

By κ-c.c. of all $\mathbb{Q}_\alpha$, $\mathbb{Q}_\alpha \lessdot \mathbb{Q}^*_\alpha := \mathbb{Q}_\alpha \cap M^*$ for each $\alpha \in M^* \cap \kappa^+$.

Let $B \subseteq V_\kappa$ code $(\mathbb{Q}^*_\alpha)_{\alpha \in M^* \cap \kappa^+}$ and the sequence of maximal antichains of $\mathbb{Q}^*_\alpha$ (for $\alpha \in M^* \cap \kappa^+$). But now, for a suitable Π^1_1 sentence ψ with B as parameter, we may find a set $C \in \mathcal{F}$ of $\delta < \kappa$ such that $(V_\delta, \in) \models \psi(B \cap V_\delta)$. With a sensible choice of ψ it follows that we may find $\delta \in C$ such that $N_i^* = Sk_{M^*}(\delta)$ satisfies (1)–(4).

Let $N_i = N_i^* \cap H(\kappa^+)$ for each i.

Let P be the satisfaction predicate for the structure

$$\langle H(\kappa^+), \in, \vec{\Phi} \rangle,$$

where $\vec{\Phi} \subseteq H(\kappa^+)$ codes $(\Phi_\alpha)_{\alpha<\kappa^+}$ in some canonical way, and let M be an elementary submodel of $H(\theta)$ containing q, $\underset{\sim}{r}_i$, $(\beta_i)_{i<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha\leq\kappa^+}$, $(N_i^*)_{i<\kappa^+}$ and P, and such that $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$.

Let $i_0 \in \kappa^+ \setminus M$. By a standard reflection argument we may find $i_1 \in \kappa^+ \cap M$ for which there exists an isomorphism

$$\Psi : (N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q) \cong (N_{i_1}, \in, P, \underset{\sim}{r}_{i_1}, \beta_{i_1}, q),$$

such that $\Psi(\xi) \leq \xi$ for every ordinal in N_{i_0}. Indeed, the existence of such an i_1 follows from the correctness of M in $H(\theta)$ about a suitable statement with parameters $(N_i)_{i<\kappa^+}$, q, P, $(\beta_i)_{i<\kappa^+}$, $(\underset{\sim}{r}_i)_{i<\kappa^+}$, and $N_{i_0} \cap M$, all of which are in M.

Let $\bar{q} = (f_q, \tau_{\bar{q}})$, where

$$\tau_{\bar{q}} = \tau_q \cup \{\langle (N_{i_0}, \beta_{i_0} + 1), (N_{i_1}, \beta_{i_1} + 1) \rangle\}$$

But $\mathbb{Q}^{N_{i_0}}_{\alpha+1} \subseteq \mathbb{Q}_{\alpha+1}$ for each α. Hence, $\mathbb{Q}_{\alpha+1} \cap N_{i_0} \lessdot \mathbb{Q}^{N_{i_0}}_{\alpha+1}$ for every $\alpha \in N^*_{i_0}$. It follows that $\bar{q} \in \mathbb{Q}_{\kappa^+}$ thanks to the choice of $N^*_{i_0}$ and $N^*_{i_1}$. We show that $\bar{q} \Vdash_{\mathbb{Q}_{\kappa^+}} \underset{\sim}{r}_{i_0} = \underset{\sim}{r}_{i_1}$.

Suppose not, and we will derive a contradiction. Thus we can find $\nu < \omega_1$ and $q' \leq_{\kappa^+} \bar{q}$ such that

$$q' \Vdash_{\mathbb{Q}_{\kappa^+}} \text{“}\nu \in \underset{\sim}{r}_{i_0} \leftrightarrow \nu \notin \underset{\sim}{r}_{i_1}\text{”}.$$

Let us assume, for concreteness, that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_0}$ and $\nu \notin \underset{\sim}{r}_{i_1}$” (the proof in the case that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_1}$ and $\nu \notin \underset{\sim}{r}_{i_0}$” is exactly the same).

By correctness of $N^*_{i_0}$ we have that this model contains a maximal antichain A of conditions in $\mathbb{Q}_{\beta_{i_0}}$ deciding the statement "$\nu \in \underset{\sim}{r}_{i_0}$". $|A| < \kappa$ by κ-c.c. Hence, since $N^*_{i_0} \cap \kappa \in \kappa$, $A \subseteq N^*_{i_0} \cap H(\kappa^+) = N_{i_0}$. Hence, we may find a common extension q'' of q' and some $r \in N_{i_0} \cap A$ such that $r \Vdash_{\mathbb{Q}_{\kappa^+}}$ "$\nu \in \underset{\sim}{r}_{i_0}$".
Also, note that, since Ψ is an isomorphism between the structures $(N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q)$ and $(N_{i_1}, \in, P, \underset{\sim}{r}_{i_1}, \beta_{i_1}, q)$, and by the choice of P, we have that

$$\Psi(r) \Vdash_{\mathbb{Q}_{\beta_{i_1}}} \text{“}\nu \in \Psi(\underset{\sim}{r}_{i_0}) = \underset{\sim}{r}_{i_1}\text{”}$$

But then, by clause (3) in the definition of condition, we have that $q'' \leq \Psi(r)$. We thus obtain that $q'' \Vdash_{\mathbb{Q}_{\kappa^+}}$ "$\nu \in \underset{\sim}{r}_{i_1}$", which is impossible as $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ "$\nu \notin \underset{\sim}{r}_{i_1}$" and $q'' \leq q'$. Contradiction. $\square$

The κ-chain condition

Some technical facts first.

Given conditions q_0, $q_1 \in \mathbb{Q}_{\kappa^+}$, we denote by

$$q_0 \oplus q_1$$

the natural amalgamation of q_0 and q_1; i.e., $q_0 \oplus q_1$ is the ordered pair (f, τ) resulting from closing q_0 and q_1 under relevant isomorphisms Ψ_{N_0,N_1} so that clause (3) in the definition of condition holds in the end.

Lemma 1 Let $\beta \leq \kappa^+$, and suppose q_0, $q_1 \in \mathbb{Q}_\beta$ are such that for every $\alpha < \beta$, if

$$(q_0 \restriction \alpha) \oplus (q_1 \restriction \alpha) \in \mathbb{Q}_\alpha,$$

then

$$(q_0 \restriction \alpha + 1) \oplus (q_1 \restriction \alpha + 1) \in \mathbb{Q}_{\alpha+1}$$

Then $q_0 \oplus q_1 \in \mathbb{Q}_\beta$.

Given an ordinal α and a set τ of edges, we will call a finite sequence $(\alpha_i)_{i<n}$ of ordinals a *τ-orbit of α* if there is a sequence $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle)_{i<n}$ of edges in τ and a sequence $(\epsilon_i)_{i<n}$ of ordinals in $\{0, 1\}$ such that

- $\alpha \in N_{\epsilon_0}^0 \cap \gamma_{\epsilon_0}^0$, and
- for each $i < n$,

$$\alpha_i = (\Psi_{N_{\epsilon_i}^i, N_{1-\epsilon_i}^i} \circ \Psi_{N_{\epsilon_{i-1}}^{i-1}, N_{1-\epsilon_{i-1}}^{i-1}} \circ \ldots \circ \Psi_{N_{\epsilon_0}^0, N_{1-\epsilon_0}^0})(\alpha)$$

  is such that $\alpha_i < \gamma_{1-\epsilon_i}^i$ and such that $\alpha_i \in N_{\epsilon_{i+1}}^{i+1} \cap \gamma_{\epsilon_{i+1}}^{i+1}$ if $i + 1 < n$.

We will call $(\alpha_i)_{i<n}$ a *descending orbit* if $\alpha_{i+1} \leq \alpha_i$ whenever $i + 1 < n$.

Lemma 2 For all $\beta \leq \kappa^+$, q_0, $q_1 \in \mathbb{Q}_\beta$ and $\alpha \in \text{dom}(f_{q_0 \oplus q_1})$ there is some $\alpha^* \in \text{dom}(f_{q_0}) \cup \text{dom}(f_{q_1})$ such that α is on some $\tau_{q_0} \cup \tau_{q_1}$–orbit of α^*. In fact, for any $\tau_{q_0 \oplus q_1}$-orbit $(\alpha_i)_{i<n}$ of an ordinal α^* and any ordinal α on $(\alpha_i)_{i<n}$ there is in fact a (possibly longer) $\tau_{q_0} \cup \tau_{q_1}$-orbit of α^* on which α is.

Given $\alpha < \kappa^+$ and given nodes x, $y \in \kappa \times \omega_1$, if $\mathbb{Q}_\alpha$ is κ-c.c., then we denote by $A^\alpha_{x,y}$ the first, in some well–order of $H(\kappa^+)$ canonically definable from Φ, maximal antichain of $\mathbb{Q}_\alpha$ consisting of conditions deciding whether or not x and y are comparable in $\underset{\sim}{T}_\alpha$.

Given $q \in \mathbb{Q}_{\kappa^+}$, we will say that q is *adequate* in case:

(1) For all nonzero α, α' in $\text{dom}(f_q)$, if $x \in \text{dom}(f_q(\alpha))$, $y \in \text{dom}(f_q(\alpha'))$, and $\mathbb{Q}_\alpha$ is κ-c.c., then $q \restriction \alpha$ extends a condition in $A^\alpha_{x,y}$.

(2) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$ and every $\alpha \in \text{dom}(f_q) \cap N_1 \cap \gamma_1$, if $\Psi_{N_1,N_0}(\alpha) < \gamma_0$, then $\Psi_{N_1,N_0}(\alpha) \in \text{dom}(f_q)$ and

$$f_q(\Psi_{N_1,N_0}(\alpha)) \restriction \delta_{N_1} \times \omega_1 = f_q(\alpha) \restriction \delta_{N_1} \times \omega_1$$

Let us call a condition *weakly adequate* if it satisfies clause (1) in the above definition.

The set of weakly adequate conditions is trivially dense. Also:

Lemma 3 Suppose q is a weakly adequate $\mathbb{Q}_{\kappa^+}$-condition, $\alpha \in \mathrm{dom}(f_q)$, and $\alpha' < \kappa^+$ is on a descending τ_q-orbit of α as witnessed by a sequence $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle)_{i \leq n}$ of edges in τ_q. Suppose $x_0 = (\rho_0, \zeta_0)$ and $x_1 = (\rho_1, \zeta_1)$ are two nodes such that

(1) $\rho_0, \rho_1 < \min\{\delta_{N_0^i} \mid i \leq n\}$, and

(2) $q \restriction \alpha$ forces that x and y are incomparable in $\underset{\sim}{T}_\alpha$.

Then $q \restriction \alpha'$ forces that x_0 and x_1 are incomparable in $\underset{\sim}{T}_{\alpha'}$.

Lemma 4 For every $\beta \leq \kappa^+$, the set of adequate $\mathbb{Q}_\beta$-conditions is dense in $\mathbb{Q}_\beta$.

We call a model Q *suitable* if Q is an elementary submodel of cardinality κ of some high enough $H(\theta)$, closed under $<\kappa$-sequences, and such that $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa^+ \rangle \in Q$. Given a suitable model Q, a bijection $\varphi : \kappa \to Q$, and an ordinal $\lambda < \kappa$, we will denote $\varphi``\lambda$ by M^φ_λ.

Lemma 4 For every $\beta \leq \kappa^+$, the set of adequate $\mathbb{Q}_\beta$-conditions is dense in $\mathbb{Q}_\beta$.

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Given $\beta \leq \kappa^+$, we will say that $\mathbb{Q}_\beta$ *has the strong* κ*-chain condition* if for every $X \in \mathcal{S}$, every suitable model Q such that $\beta, X \in Q$, every bijection $\varphi : \kappa \to Q$, and every two sequences

$$(q^0_\lambda \mid \lambda \in X) \in Q$$

and

$$(q^1_\lambda \mid \lambda \in X) \in Q$$

of $\mathbb{Q}_\beta$-conditions, if $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$ for every $\lambda \in X$, then there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences

$$(q^{00}_\lambda \mid \lambda \in Y)$$

and

$$(q^{11}_\lambda \mid \lambda \in Y)$$

of $\mathbb{Q}_\beta$-conditions with the following properties.

(1) $q^{00}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q^{11}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.

(2) For all $\lambda < \lambda^*$ in Y, $q^{00}_\lambda \oplus q^{11}_{\lambda^*}$ is a common extension of q^{00}_λ and $q^{11}_{\lambda^*}$.

Given a suitable model Q such that $\beta \in Q$, a bijection $\varphi : \kappa \to Q$, a $\mathbb{Q}_\beta$-condition $q \in Q$, and $\lambda < \kappa$, let us say that q *is λ-compatible with respect to φ and β* if, letting $\mathbb{Q}^*_\beta = \mathbb{Q}_\beta \cap Q$, we have that

- $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda \lessdot \mathbb{Q}^*_\beta$,
- $q \restriction M^\varphi_\lambda \in \mathbb{Q}^*_\beta$, and
- $q \restriction M^\varphi_\lambda$ forces in $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda$ that q is in the quotient forcing $\mathbb{Q}^*_\beta / \dot{G}_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda}$; equivalently, for every $r \leq_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda} q \restriction M^\varphi_\lambda$, r is compatible with q.

Rather than proving that every $\mathbb{Q}_\beta$ has the κ-c.c., we prove the following more informative lemma.

Lemma

The following holds for every $\beta \leq \kappa^+$.

$(1)_\beta$ $\mathbb{Q}_\beta$ *has the strong* κ*-chain condition.*

$(2)_\beta$ *Suppose* $D \in \mathcal{F}$, Q *is a suitable model,* β, $D \in Q$, $\varphi : \kappa \to Q$ *is a bijection, and* $(q^0_\lambda \mid \lambda \in D) \in Q$ *and* $(q^1_\lambda \mid \lambda \in D) \in Q$ *are sequences of adequate* $\mathbb{Q}_\beta$*-conditions. Then there is some* $D' \in \mathcal{F}$ *such that* $D' \subseteq D$ *and such that for every* $\lambda \in D'$*, if* $q^0_\lambda \upharpoonright M^\varphi_\lambda = q^1_\lambda \upharpoonright M^\varphi_\lambda$*, then there are conditions* $q'^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ *and* $q'^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ *such that*

(a) $q'^0_\lambda \upharpoonright M^\varphi_\lambda = q'^1_\lambda \upharpoonright M^\varphi_\lambda$ *and*

(b) q'^0_λ *and* q'^1_λ *are both* λ*-compatible with respect to* φ *and* β.

The proof is by induction on β. Let $\beta \leq \kappa^+$ and suppose $(1)_\alpha$ and $(2)_\alpha$ holds for all $\alpha < \beta$. We will show that $(1)_\beta$ and $(2)_\beta$ also hold.

There is nothing to prove for $\beta = 0$, the case $\beta = 1$ is trivial using the inaccessibility of κ, and the case $\beta = \kappa^+$ follows form $\mathbb{Q}_{\kappa^+} = \bigcup_{\beta < \kappa^+} \mathbb{Q}_\beta$. Hence, let us assume $1 < \beta < \kappa^+$.

Suppose next that $\beta < \kappa^+$. We start with the proof of $(1)_\beta$.

Let Q, φ, $X \in \mathcal{S}$ and $(q^0_\lambda \mid \lambda \in X)$ and $(q^1_\lambda \mid \lambda \in X)$ be as in the definition of strong κ-c.c. In what follows, we will write M_λ instead of M^φ_λ.

The proof is by induction on β. Let $\beta \leq \kappa^+$ and suppose $(1)_\alpha$ and $(2)_\alpha$ holds for all $\alpha < \beta$. We will show that $(1)_\beta$ and $(2)_\beta$ also hold.

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Suppose next that $\beta < \kappa^+$. We start with the proof of $(1)_\beta$.

Let Q, φ, $X \in \mathcal{S}$ and $(q^0_\lambda \mid \lambda \in X)$ and $(q^1_\lambda \mid \lambda \in X)$ be as in the definition of strong κ-c.c. In what follows, we will write M_λ instead of M^φ_λ.

Let $\mathbb{Q}^*_\alpha = \mathbb{Q}_\alpha \cap Q$ for every $\alpha \in Q \cap (\beta + 1)$. By the induction hypothesis, $\mathbb{Q}_\alpha$ has the κ-c.c. for every $\alpha \in Q \cap \beta$. Hence, since ${}^{<\kappa}Q \subseteq Q$, we have that $\mathbb{Q}^*_\alpha \lessdot \mathbb{Q}_\alpha$ for every such α; in particular, we have that for every $\alpha \in Q \cap \beta$, $\mathbb{Q}^*_\alpha$ forces over V that $\underset{\sim}{T}_\alpha$ does not have κ-branches.

Given a nonzero $\alpha \in \beta$, a node $x = (\rho, \zeta)$ and an ordinal $\bar{\rho} < \rho$, let $B^\alpha_{x,\bar{\rho}}$ denote the least, in some well–order of $H(\kappa^+)$ canonically defined from Φ, maximal antichain of $\mathbb{Q}_\alpha$ consisting of conditions deciding some $\bar{\zeta} < \omega_1$ such that the node $\bar{x} = (\bar{\rho}, \bar{\zeta})$ is below x in $\underset{\sim}{T}_\alpha$.

Given

- conditions q^0, q^1,
- $\alpha \in \mathrm{dom}(f_{q^0})$ and $\alpha' \in \mathrm{dom}(f_{q^1})$,
- nodes $x = (\rho_0, \zeta_0)$ and $y = (\rho_1, \zeta_1)$ such that $x \in \mathrm{dom}(f_{q^0}(\alpha))$ and $y \in \mathrm{dom}(f_{q^1}(\alpha'))$,[1] and
- $\lambda < \kappa$,

we will say that *x and y are separated below λ at stages α and α' by $q^1 \restriction \alpha$ and $q^0 \restriction \alpha'$ (via $\bar{x}$, $\bar{y}$)* if there are $\bar{\rho} < \lambda$ and $\zeta \neq \zeta'$ in ω_1 such that $\bar{x} = (\bar{\rho}, \zeta)$, $\bar{y} = (\bar{\rho}, \zeta')$, and such that

(1) $q^0 \restriction \alpha$ extends a condition in $B^{\alpha}_{x,\bar{\rho}}$ forcing $\bar{x}$ to be below x in $\underset{\sim}{T}_{\alpha}$ and

(2) $q^1 \restriction \alpha'$ extends a condition in $B^{\alpha'}_{y,\bar{\rho}}$ forcing $\bar{y}$ to be below y in $\underset{\sim}{T}_{\alpha'}$.

[1] α and α' may or may not be equal, and the same applies to x and y.

Given $Y \in \mathcal{S}$ such that $Y \subseteq X$ and such that $M_\lambda \prec Q$, $M_\lambda \cap \kappa = \lambda$, and ${}^{<\lambda}M_\lambda \subseteq M_\lambda$ for all $\lambda \in Y$, and given two sequences $\sigma^{00} = (q_\lambda^{00} \mid \lambda \in Y)$, $\sigma^{11} = (q_\lambda^{11} \mid \lambda \in Y)$ of adequate $\mathbb{Q}_\beta^*$-conditions, we say that σ^{00}, σ^{11} *is a separating pair for* σ^0 *and* σ^1 if the following holds.

(1) $q_\lambda^{00} \leq_{\mathbb{Q}_\beta} q_\lambda^0$ and $q_\lambda^{11} \leq_{\mathbb{Q}_\beta} q_\lambda^1$ for all $\lambda \in Y$.

(2) For all $\lambda \in Y$, all nonzero $\alpha \in \mathrm{dom}(f_{q_\lambda^{00}}) \cap M_\lambda$ and $\alpha' \in \mathrm{dom}(f_{q_\lambda^{11}})$ such that $\alpha' \leq \alpha$, and all $x \in \mathrm{dom}(f_{q_\lambda^{00}}(\alpha)) \setminus (\lambda \times \omega_1)$ and $y \in \mathrm{dom}(f_{q_\lambda^{11}}(\alpha')) \setminus (\lambda \times \omega_1)$, x and y are separated below λ at stages α and α' by $q_\lambda^{00} \restriction \alpha$ and $q_\lambda^{11} \restriction \alpha'$ via some pair $\chi_0(x, y, \alpha, \alpha', \lambda)$, $\chi_1(x, y, \alpha, \alpha', \lambda)$.

(3) The following holds for all $\lambda_0 < \lambda_1$ in Y.

(a) $q^{00}_{\lambda_0} \restriction M_{\lambda_0} = q^{11}_{\lambda_1} \restriction M_{\lambda_1}$

(b) $\mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cap M_{\lambda_0} = \mathrm{dom}(f_{q^{11}_{\lambda_1}}) \cap M_{\lambda_1}$

(c) $q^{00}_{\lambda_0} \in M_{\lambda_1}$

(d) For some ordinal ς,

$$\sup(R_{\lambda_0} \cup \Delta_{\lambda_0}) = \sup(R_{\lambda_1} \cup \Delta_{\lambda_1}) = \varsigma$$

where, for every $\epsilon \in \{0, 1\}$,

$$R_\epsilon = \{\rho < \lambda_\epsilon \mid \alpha \in \mathrm{dom}(f_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}), (\rho, \zeta) \in \mathrm{dom}(f_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}(\alpha))\}$$

and

$$\Delta_\epsilon = \{\delta_{N_0} < \lambda_\epsilon \mid \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}\}$$

(e) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q^{11}_{\lambda_1}}$, if $\delta_{N_0} < \lambda_1$, then $(N_0 \cup N_1) \cap (\mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cup \tau_{q^{00}_{\lambda_0}}) \subseteq M_{\lambda_0}$.

(4) For all $\lambda_0 < \lambda_1$ in X, all nonzero $\alpha \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cap M_{\lambda_0}$ and $\alpha' \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$ such that $\alpha' \leq \alpha$, and all nodes

$$x \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha)) \setminus (\lambda_0 \times \omega_1)$$

and

$$y' \in \mathrm{dom}(f_{q^{11}_{\lambda_1}(\alpha')}) \setminus (\lambda_1 \times \omega_1)$$

there are

- a node $x' \in \mathrm{dom}(f_{q^{00}_{\lambda_1}(\alpha)}) \setminus (\lambda_1 \times \omega_1)$,
- a stage $\alpha^\dagger \in \mathrm{dom}(f_{q^{11}_{\lambda_0}})$ such that $\alpha^\dagger \leq \alpha$, and
- a node $y \in \mathrm{dom}(f_{q^{11}_{\lambda_0}(\alpha^\dagger)}) \setminus (\lambda_0 \times \omega_1)$

such that

$$\chi_0(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_0(x', y', \alpha, \alpha', \lambda_1)$$

and

$$\chi_1(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_1(x', y', \alpha, \alpha', \lambda_1)$$

Claim 1: Let $Y \in \mathcal{S}$ be such that $M_\lambda \cap \kappa = \lambda$ for all $\lambda \in Y$ and suppose $\sigma^{00} = (q^{00}_\lambda \mid \lambda \in Y)$, $\sigma^{11} = (q^{11}_\lambda \mid \lambda \in Y)$ is a separating pair for σ^0 and σ^1. Then for all $\lambda_0 < \lambda < \lambda_1$ in Y, $q^{00}_{\lambda_0} \oplus q^{11}_{\lambda_1}$ is a common extension of $q^{00}_{\lambda_0}$ and $q^{11}_{\lambda_1}$ in $\mathbb{Q}_\beta$.

The following is essentially due to Laver–Shelah.

Claim 1: Let $Y \in \mathcal{S}$ be such that $M_\lambda \cap \kappa = \lambda$ for all $\lambda \in Y$ and suppose $\sigma^{00} = (q^{00}_\lambda \mid \lambda \in Y)$, $\sigma^{11} = (q^{11}_\lambda \mid \lambda \in Y)$ is a separating pair for σ^0 and σ^1. Then for all $\lambda_0 < \lambda < \lambda_1$ in Y, $q^{00}_{\lambda_0} \oplus q^{11}_{\lambda_1}$ is a common extension of $q^{00}_{\lambda_0}$ and $q^{11}_{\lambda_1}$ in $\mathbb{Q}_\beta$.

The following is essentially due to Laver–Shelah.

Claim 2: Suppose $Z \in \mathcal{S}$, $(p^0_\lambda \mid \lambda \in Z) \in Q$ and $(p^1_\lambda \mid \lambda \in Z) \in Q$ are sequences of conditions in $\mathbb{Q}^*_\beta$ such that $p^0_\lambda \restriction M_\lambda$ and $p^1_\lambda \restriction M_\lambda$ are compatible conditions in $\mathbb{Q}^*_\beta \cap M_\lambda$ for every $\lambda \in Z$, and suppose that for every $\lambda \in Z$,

- p^0_λ and p^1_λ are λ-compatible with respect to φ and α for all $\alpha \in \beta \cap Q$,
- $\alpha_\lambda \in \mathrm{dom}(f_{p^0_\lambda}) \cap M_\lambda$,
- $\alpha'_\lambda \in \mathrm{dom}(f_{p^1_\lambda}) \cap M_\lambda$ is a nonzero ordinal such that $\alpha'_\lambda \leq \alpha_\lambda$, and
- $x_\lambda = (\rho^0_\lambda, \zeta^0_\lambda)$ and $y_\lambda = (\rho^1_\lambda, \zeta^1_\lambda)$ are nodes in $(\kappa \setminus \lambda) \times \omega_1$ such that $x_\lambda \in \mathrm{dom}(f_{p^0_\lambda}(\alpha_\lambda))$ and $y_\lambda \in \mathrm{dom}(f_{p^1_\lambda}(\alpha'_\lambda))$.

Then there is $D \in \mathcal{F}$, together with two sequences $(p^2_\lambda \mid \lambda \in Z \cap D)$, $(p^3_\lambda \mid \lambda \in Z \cap D)$ of conditions in $\mathbb{Q}^*_\beta$ such that

(1) for each $\lambda \in Z \cap D$, $p^2_\lambda \leq q^0_\lambda$ and $p^3_\lambda \leq p^1_\lambda$,

(2) for each $\lambda \in Z \cap D$, $p^2_\lambda \restriction M_\lambda$ and $p^3_\lambda \restriction M_\lambda$ are compatible in $\mathbb{Q}^*_\beta \cap M_\lambda$, and

(3) for each $\lambda \in Z \cap D$, x_λ and y_λ are separated below λ at stages α_λ and α'_λ by $p^2_\lambda \restriction \alpha_\lambda$ and $p^3_\lambda \restriction \alpha'_\lambda$.

By Claim 1, in order to conclude the proof of current instance of $(1)_\beta$, it suffices to prove:

Claim 3: There is a separating pair for σ^0 and σ^1.

Claim 3 is proved by a construction in countably many steps using Claim 2, and a pressing–down argument using the normality of $\mathcal{F}$.

We are left with proving $(2)_\beta$. But this is established with essentially the same argument as in the corresponding proof in the Laver–Shelah paper.

This concludes the proof of the κ-chain condition lemma and the proof of the theorem.

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This concludes the proof of the κ-chain condition lemma and the proof of the theorem.

An open question

Question (Shelah): Is it consistent to have GCH together with a successor cardinal $\kappa \geq \omega_1$ such that all κ-Aronszajn and all κ^+-Aronszajn trees are special?