$\ll$-Increasing Approximate Units in C*-Algebras

Tristan Bice
joint work with **Piotr Koszmider**

Institute of Mathematics of the Polish Academy of Sciences

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Commutative C*-algebras $\approx$ $C_0(X)$.

- Note X is compact iff $C_0(X) = C(X)$ iff $1 \in C_0(X)$.
- e.g. for $X = \mathbb{N}, \mathbb{R}, \mathbb{R}^2$, etc. $C_0(X)$ is not unital.

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- Proof: $A^{<1}_+ = \{a \in A_+ : ||a|| < 1\}$ is $\leq$-directed. $\square$

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- $\therefore$ separable C*-algebras have $\ll$-increasing approximate units.
- C*-algebra theory often requires separability partly $\because$ of this.

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- But $p_n, q_n \ll e_n \in (M_2)^1_+$ implies $e_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $(e_n) \notin D$.

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- Note $p = (p_n)$ and $q = (q_n)$ are projections so $p, q \in D_c$.
- But $p_n, q_n \ll e_n \in (M_2)^1_+$ implies $e_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $(e_n) \notin D$.
- i.e. p and q have no upper bound in D_c or even D^1_+.

General Compact Supports

- For commutative A, the compactly supported elements are
$$A_c = \{a \in A^1_+ : \exists b \in A^1_+ \ (a \ll b)\}.$$
- Could we extend the commutative case directly via A_c?
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Given $\epsilon > 0$ find $\delta > 0$ such that, for $u, v \in A^1_+$,

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- e.g. need Pedersen's non-commutative Riesz decomposition.

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- Proof uses the combinatorics of the tree to find a coordinate t such that any hypothetical $\ll$-unit would bound p and q at t.

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- PFA $\Rightarrow$ there are no Canadian trees.

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$S \subseteq \mathcal{P}(\mathbb{N})$ with $|S| = \omega_1$ is separated if every partition $S = T \cup U$ comes from a partition $\mathbb{N} = Y \cup Z$, i.e.

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Corollary (B.-Koszmider 2017)

"All C*-subalgebras of $\mathcal{B}(\ell^2)$ have $\ll$-units" is independent of ZFC.

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Question

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