Some reflection principles at large continuum

André Ottenbreit Maschio Rodrigues
a joint work with Sakaé Fuchino and Hiroshi Sakai

Kobe University

November 8, 2018

1 Stationary Logic and reflection principles

2 Consistency Result

Definition (WRP)

For every regular $\eta \geq \aleph_2$*, every stationary* $S \subseteq [\eta]^{\aleph_0}$ *and every* $X \in [\eta]^{\aleph_1}$*, there is* $Y \in [\eta]^{\aleph_1}$ *such that*

1. $X \subseteq Y$*;*
2. $S \cap [Y]^{\aleph_0}$ *is stationary in* $[Y]^{\aleph_0}$*.*

(in Jech's book, this principle is called just RP)

WRP imposes the following boundary for the size of the continuum:

Theorem (Todorčević)

WRP implies $2^{\aleph_0} \leq \aleph_2$*.*

Weak Reflection Principle

Definition (WRP)

For every regular $\eta \geq \aleph_2$*, every stationary* $S \subseteq [\eta]^{\aleph_0}$ *and every* $X \in [\eta]^{\aleph_1}$*, there is* $Y \in [\eta]^{\aleph_1}$ *such that*

1. $X \subseteq Y$;
2. $S \cap [Y]^{\aleph_0}$ *is stationary in* $[Y]^{\aleph_0}$.

(in Jech's book, this principle is called just RP)
WRP imposes the following boundary for the size of the continuum:

Theorem (Todorčević)

WRP implies $2^{\aleph_0} \leq \aleph_2$.

- WRP is consistent with CH, because it holds if we Levy collapse a supercompact cardinal to $\aleph_2$.
- WRP is also compatible with $\neg$CH since WRP follows from MM, and MM$\Rightarrow 2^{\aleph_0} = \aleph_2$.
- Now we present a characterization of WRP:

Lemma

WRP is equivalent to the following statement:
For any uncountable cardinal η, any stationary $S \subseteq [\mathcal{H}(\eta)]^{\aleph_0}$ and any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\eta)]^{\aleph_1}$ such that

1. $\mathfrak{A}{\restriction}_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$.

- WRP is consistent with CH, because it holds if we Levy collapse a supercompact cardinal to $\aleph_2$.
- WRP is also compatible with $\neg$CH since WRP follows from MM, and MM$\Rightarrow 2^{\aleph_0} = \aleph_2$.
- Now we present a characterization of WRP:

Lemma

WRP is equivalent to the following statement:
For any uncountable cardinal η, any stationary $S \subseteq [\mathcal{H}(\eta)]^{\aleph_0}$ and any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\eta)]^{\aleph_1}$ such that

1. $\mathfrak{A}\restriction_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$.

- WRP is consistent with CH, because it holds if we Levy collapse a supercompact cardinal to $\aleph_2$.
- WRP is also compatible with $\neg$CH since WRP follows from MM, and MM$\Rightarrow 2^{\aleph_0} = \aleph_2$.
- Now we present a characterization of WRP:

Lemma

WRP is equivalent to the following statement:
For any uncountable cardinal η, any stationary $S \subseteq [\mathcal{H}(\eta)]^{\aleph_0}$ and any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\eta)]^{\aleph_1}$ such that

1. $\mathfrak{A}\restriction_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$.

- WRP is consistent with CH, because it holds if we Levy collapse a supercompact cardinal to $\aleph_2$.
- WRP is also compatible with $\neg$CH since WRP follows from MM, and MM$\Rightarrow 2^{\aleph_0} = \aleph_2$.
- Now we present a characterization of WRP:

Lemma

WRP is equivalent to the following statement:
For any uncountable cardinal η, any stationary $S \subseteq [\mathcal{H}(\eta)]^{\aleph_0}$ and any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\eta)]^{\aleph_1}$ such that

1. $\mathfrak{A}\restriction_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$.

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0, \text{II}}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0, \mathrm{II}}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0, II}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Stationary Logics-1

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0, II}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0, \mathrm{II}}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Define the weak second order logic $\mathcal{L}^{\aleph_0}$ as follows:

- first order variables (lowercase letters) $x, y, z, \ldots$;
- weak second order variables (capital letters) $X, Y, Z, \ldots$ to be interpreted as countable subsets of the underlying set of a structure;
- first order quantifiers $\forall x, \exists x$;
- we introduce in this logic the symbol "ε": $x \varepsilon X$ shall be interpreted as $x \in X$, and it must be used with a first and a second order variable respectively.

Define also $\mathcal{L}^{\aleph_0,\mathrm{II}}$ by adding the second order quantifiers $\forall X, \exists Y$ to $\mathcal{L}^{\aleph_0}$.

Now define $\mathcal{L}^{\aleph_0}_{\text{stat}}$ by adding to $\mathcal{L}^{\aleph_0}$ a new quantifier "stat$X$" for second order variables to be interpreted as follows:
Let φ be an $\mathcal{L}^{\aleph_0}_{\text{stat}}$-formula. stat$X\varphi(X)$ means that φ holds for stationary many X, i.e. given a structure $\mathfrak{A} = \langle A, \ldots \rangle$ we define

$$\mathfrak{A} \models \text{“}\operatorname{stat} X\varphi(X)\text{”}$$
$$\Updownarrow$$
$$\{B \in [A]^{\aleph_0} : \mathfrak{A} \models \text{“}\varphi(B)\text{”}\} \text{ is stationary in } [A]^{\aleph_0}$$

In $\mathcal{L}^{\aleph_0}_{\text{stat}}$ we can also define aa$X$ (for almost all X) the dual quantifier for stat$X$. aa$X\varphi(X)$ is an abbreviation for $\neg$stat$X\neg\varphi(X)$. In other words:

$$\mathfrak{A} \models \text{“}\operatorname{aa} X\varphi(X)\text{”}$$
$$\Updownarrow$$
$$\{B \in [A]^{\aleph_0} : \mathfrak{A} \models \text{“}\varphi(B)\text{”}\} \text{ contains a club subset of } [A]^{\aleph_0}$$

Now define $\mathcal{L}^{\aleph_0}_{\text{stat}}$ by adding to $\mathcal{L}^{\aleph_0}$ a new quantifier "stat$X$" for second order variables to be interpreted as follows:
Let φ be an $\mathcal{L}^{\aleph_0}_{\text{stat}}$-formula. stat$X\varphi(X)$ means that φ holds for stationary many X, i.e. given a structure $\mathfrak{A} = \langle A, \ldots \rangle$ we define

$$\mathfrak{A} \models \text{“ stat}X\varphi(X)\text{”}$$
$$\Updownarrow$$
$$\{B \in [A]^{\aleph_0} \ : \ \mathfrak{A} \models \text{“ }\varphi(B)\text{”}\} \text{ is stationary in } [A]^{\aleph_0}$$

In $\mathcal{L}^{\aleph_0}_{\text{stat}}$ we can also define aa$X$ (for almost all X) the dual quantifier for stat$X$. aa$X\varphi(X)$ is an abbreviation for $\neg$stat$X\neg\varphi(X)$. In other words:

$$\mathfrak{A} \models \text{“ aa}X\varphi(X)\text{”}$$
$$\Updownarrow$$
$$\{B \in [A]^{\aleph_0} \ : \ \mathfrak{A} \models \text{“ }\varphi(B)\text{”}\} \text{ contains a club subset of } [A]^{\aleph_0}$$

$\mathcal{L}$-elementary substructure

Let $\mathcal{L}$ be a logic (or family of formulas in a logic), structures $\mathfrak{A}, \mathfrak{B}$ in the same signature, $\mathfrak{B} \subseteq \mathfrak{A}$. We say that

$$\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$$

($\mathfrak{B}$ is an *$\mathcal{L}$-elementary substructure* of $\mathfrak{A}$) iff: for all formulas $\varphi(x_0, \ldots, x_n, X_0, \ldots, X_m)$ in $\mathcal{L}$, for all $b_0, \ldots, b_n$ first order objects of $\mathfrak{B}$, all $B_0, \ldots, B_m$ second order objects of $\mathfrak{B}$, we have

$$\mathfrak{B} \models \text{“} \varphi(b_0, \ldots, B_0, \ldots) \text{”} \Leftrightarrow \mathfrak{A} \models \text{“} \varphi(b_0, \ldots, B_0, \ldots) \text{”}$$

Similarly, we write

$$\mathfrak{A} \prec_{\mathcal{L}}^{-} \mathfrak{B}$$

iff for all formulas φ in $\mathcal{L}$ which have only first order free variables and for all $b_0, \ldots b_n$ first order objects in $\mathfrak{B}$, we have

$$\mathfrak{B} \models \text{``}\varphi(b_0, \ldots, b_n)\text{''} \Leftrightarrow \mathfrak{A} \models \text{``}\varphi(b_0, \ldots, b_n)\text{''}$$

Strong Downwards Löwenheim-Skolem reflection

Let $\mathcal{L}$ be a logic and μ an infinite cardinal. Define:

$\mathrm{SDLS}(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ of cardinality $< \mu$.

Similarly, define:

$\mathrm{SDLS}^{-}(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}}^{-} \mathfrak{A}$ of cardinality $< \mu$.

This gives us a variety of different reflection statements.

Strong Downwards Löwenheim-Skolem reflection

Let $\mathcal{L}$ be a logic and μ an infinite cardinal. Define:

$\mathsf{SDLS}(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ of cardinality $< \mu$.

Similarly, define:

$\mathsf{SDLS}^{-}(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}}^{-} \mathfrak{A}$ of cardinality $< \mu$.

This gives us a variety of different reflection statements.

Strong Downwards Löwenheim-Skolem reflection

Let $\mathcal{L}$ be a logic and μ an infinite cardinal. Define:

SDLS$(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ of cardinality $< \mu$.

Similarly, define:

SDLS$^{-}(\mathcal{L}, < \mu)$

For any structure $\mathfrak{A}$ of countable signature of cardinality $\geq \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}}^{-} \mathfrak{A}$ of cardinality $< \mu$.

This gives us a variety of different reflection statements.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle : n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X (X \subset \omega \to \exists x \forall n \in \omega (n \varepsilon X \leftrightarrow n E x))$$

Clearly $\mathfrak{A} \models$ "ψ". By SDLS$^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$, there is some $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0, II}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0,II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle : n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X (X \subset \omega \rightarrow \exists x \forall n \in \omega (n \varepsilon X \leftrightarrow nEx))$$

Clearly $\mathfrak{A} \models$ "ψ". By SDLS$^{-}(\mathcal{L}^{\aleph_0,\mathrm{II}}, < \aleph_2)$, there is some $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle : n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X (X \subset \omega \to \exists x \forall n \in \omega (n \varepsilon X \leftrightarrow n E x))$$

Clearly $\mathfrak{A} \models$ "ψ". By SDLS$^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$, there is some $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0, II}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle$, $|B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^-(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^-(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle \ : \ n \in a \subseteq \omega\}$.
Consider also the formula

$$\psi = \forall X (X \subset \omega \to \exists x \forall n \in \omega (n \varepsilon X \leftrightarrow nEx))$$

Clearly $\mathfrak{A} \models$ "ψ". By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0, \mathrm{II}}, < \aleph_2)$, there is some $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0, \mathrm{II}}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ".
Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^-(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^-(\mathcal{L}^{\aleph_0,II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle \ : \ n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X(X \subset \omega \to \exists x \forall n \in \omega(n \varepsilon X \leftrightarrow nEx))$$

Clearly $\mathfrak{A} \models$ "ψ". By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0,\mathrm{II}}, < \aleph_2)$, there is some $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^-(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^-(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle \ : \ n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X(X \subset \omega \to \exists x \forall n \in \omega(n \varepsilon X \leftrightarrow nEx))$$

Clearly $\mathfrak{A} \models$ "ψ". By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0, \mathrm{II}}, < \aleph_2)$, there is some $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0, \mathrm{II}}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ "ψ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Example

$SDLS^-(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the usual Downward Löwenheim-Skolem Theorem for the first order logic, and thus holds in ZFC.

Example

$SDLS^-(\mathcal{L}^{\aleph_0,II}, < \aleph_2)$ implies CH.

Rough idea: we can use the following "trick" to code second order objects into first order objects. Consider the structure $\mathfrak{A} = \langle \omega \cup \mathcal{P}(\omega), \omega, E \rangle$, where $E = \{\langle n, a \rangle \ : \ n \in a \subseteq \omega\}$. Consider also the formula

$$\psi = \forall X(X \subset \omega \to \exists x \forall n \in \omega(n \varepsilon X \leftrightarrow nEx))$$

Clearly $\mathfrak{A} \models$ " ψ ". By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0,\mathrm{II}}, < \aleph_2)$, there is some $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \mathfrak{A}$ ($\mathfrak{B} = \langle B, \ldots \rangle, |B| < \aleph_2$) such that $\mathfrak{B} \models$ " ψ ". Then $\mathcal{P}(\omega) \subseteq B$, thus $2^{\aleph_0} < \aleph_2$.

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. *SDLS*$(\mathcal{L}^{\aleph_0}, < \aleph_2)$*;*
3. *SDLS*$^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*
4. *SDLS*$(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*

Lemma

- *(Magidor, 2016) SDLS*$^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies FRP;*
- *SDLS*$^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies WRP.*
- *GRP*$^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies SDLS*$(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa)$*.*

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. *SDLS*$(\mathcal{L}^{\aleph_0}, < \aleph_2)$*;*
3. *SDLS*$^-(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*
4. *SDLS*$(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*

Lemma

- *(Magidor, 2016) SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies FRP;*
- *SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies WRP.*
- $GRP^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies SDLS*$(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa)$.

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. *SDLS*$(\mathcal{L}^{\aleph_0}, <\aleph_2)$*;*
3. *SDLS*$^-(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$*;*
4. *SDLS*$(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$*;*

Lemma

- *(Magidor, 2016) SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies FRP;*
- *SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies WRP.*
- *GRP*$^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies SDLS*$(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$.

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. *SDLS*$(\mathcal{L}^{\aleph_0}, < \aleph_2)$*;*
3. *SDLS*$^-(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*
4. *SDLS*$(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$*;*

Lemma

- *(Magidor, 2016) SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies FRP;*
- *SDLS*$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies WRP.*
- $GRP^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies SDLS*$(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa)$*.*

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. $SDLS(\mathcal{L}^{\aleph_0}, <\aleph_2)$;
3. $SDLS^{-}(\mathcal{L}^{\aleph_0, II}, <\aleph_2)$;
4. $SDLS(\mathcal{L}^{\aleph_0, II}, <\aleph_2)$;

Lemma

- *(Magidor, 2016)* $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies FRP;*
- $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies WRP.*
- $GRP^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies* $SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, <\kappa)$.

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. $SDLS(\mathcal{L}^{\aleph_0}, < \aleph_2)$;
3. $SDLS^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$;
4. $SDLS(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$;

Lemma

- *(Magidor, 2016)* $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies FRP;*
- $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies WRP.*
- $GRP^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies* $SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa)$.

Actually, we have much more:

Lemma

In ZFC, the following are equivalent:

1. *CH;*
2. $SDLS(\mathcal{L}^{\aleph_0}, < \aleph_2)$;
3. $SDLS^{-}(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$;
4. $SDLS(\mathcal{L}^{\aleph_0, II}, < \aleph_2)$;

Lemma

- *(Magidor, 2016)* $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies FRP;*
- $SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies WRP.*
- $GRP^{\omega}_{<\kappa}$ *(Game Reflection Principle by Bernard König) implies* $SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa)$.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ and the continuum

Therefore we have

$$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) \Rightarrow \mathsf{WRP} \Rightarrow 2^{\aleph_0} \leq \aleph_2$$

Similarly to WRP, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is compatible with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$.

Furthermore, for $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ imposes a stricter restriction to the size of the continuum:

Lemma

For any $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

In particular, since $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2) \Leftrightarrow \mathsf{CH}$, we have :

Corollary

For any $\kappa \geq \aleph_2$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ and the continuum

Therefore we have

$$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) \Rightarrow \mathsf{WRP} \Rightarrow 2^{\aleph_0} \leq \aleph_2$$

Similarly to WRP, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is compatible with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$.

Furthermore, for $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ imposes a stricter restriction to the size of the continuum:

Lemma

For any $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

In particular, since $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2) \Leftrightarrow \mathsf{CH}$, we have :

Corollary

For any $\kappa \geq \aleph_2$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \kappa)$ and the continuum

Therefore we have

$$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \aleph_2) \Rightarrow \mathsf{WRP} \Rightarrow 2^{\aleph_0} \leq \aleph_2$$

Similarly to WRP, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \aleph_2)$ is compatible with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$.
Furthermore, for $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \kappa)$ imposes a stricter restriction to the size of the continuum:

Lemma

For any $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

In particular, since $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2) \Leftrightarrow \mathsf{CH}$, we have :

Corollary

For any $\kappa \geq \aleph_2$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ and the continuum

Therefore we have

$$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) \Rightarrow \mathsf{WRP} \Rightarrow 2^{\aleph_0} \leq \aleph_2$$

Similarly to WRP, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is compatible with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$.
Furthermore, for $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ imposes a stricter restriction to the size of the continuum:

Lemma

For any $\kappa > \aleph_2$, SDLS$^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

In particular, since $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2) \Leftrightarrow$ CH, we have :

Corollary

For any $\kappa \geq \aleph_2$, SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ and the continuum

Therefore we have

$$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) \Rightarrow \mathsf{WRP} \Rightarrow 2^{\aleph_0} \leq \aleph_2$$

Similarly to WRP, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is compatible with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$.
Furthermore, for $\kappa > \aleph_2$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ imposes a stricter restriction to the size of the continuum:

Lemma

For any $\kappa > \aleph_2$, $SDLS^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

In particular, since $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2) \Leftrightarrow \mathsf{CH}$, we have :

Corollary

For any $\kappa \geq \aleph_2$, $SDLS(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ implies $2^{\aleph_0} < \kappa$.

Recall the previous characterization of WRP:

WRP equivalent to:

For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and structure $\mathfrak{A} = \langle \mathcal{H}(\lambda), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ such that

1. $\mathfrak{A} \restriction_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

We now present some reflection statements which can also characterize some of the SLDS statements.

Recall the previous characterization of WRP:

WRP equivalent to:

For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and structure $\mathfrak{A} = \langle \mathcal{H}(\lambda), \in, \ldots \rangle$ in signature of size $\leq \aleph_1$, there is $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ such that

1. $\mathfrak{A} \restriction_M \prec \mathfrak{A}$;
2. $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

We now present some reflection statements which can also characterize some of the SLDS statements.

Let $\kappa \geq \aleph_2$ be a regular cardinal. Define:

$(*)^{-}_{<\kappa}$

Given any $\eta > \kappa$, for any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in countable signature and any family $S = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ satisfying:

1. $\mathfrak{A}\restriction_N \prec \mathfrak{A}$;
2. for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is an stationary subset of $[N]^{\aleph_0}$.

Let $\kappa \geq \aleph_2$ be a regular cardinal. Define:

$(*)_{<\kappa}$

Given any $\eta > \kappa$, for any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in countable signature and any family $S = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ satisfying:

0. N is internally club, i.e., N contains a club subset of $[N]^{\aleph_0}$;
1. $\mathfrak{A} \restriction N \prec \mathfrak{A}$;
2. for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is an stationary subset of $[N]^{\aleph_0}$.

Let $\kappa \geq \aleph_2$ be a regular cardinal. Define:

$(*)^+_{<\kappa}$

Given any $\eta > \kappa$, for any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in countable signature and any family $S = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ satisfying:

0. $[N]^{\aleph_0} \subseteq N$;
1. $\mathfrak{A} \restriction N \prec \mathfrak{A}$;
2. for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is an stationary subset of $[N]^{\aleph_0}$.

Clearly $(*)^+_{<\kappa} \Rightarrow (*)_{<\kappa} \Rightarrow (*)^-_{<\kappa}$.
Similarly to WRP, for any regular $\kappa \geq \aleph_2$, the consistency of $(*)^+_{<\kappa^+}$ (denote by $(*)^+_{\leq\kappa}$) can be obtained by Levy collapsing a supercompact cardinal bigger than κ to become κ^+.

Let $\kappa \geq \aleph_2$ be a regular cardinal. Define:

$(*)^+_{<\kappa}$

Given any $\eta > \kappa$, for any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in countable signature and any family $S = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ satisfying:

0. $[N]^{\aleph_0} \subseteq N$;
1. $\mathfrak{A} \restriction_N \prec \mathfrak{A}$;
2. for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is an stationary subset of $[N]^{\aleph_0}$.

Clearly $(*)^+_{<\kappa} \Rightarrow (*)_{<\kappa} \Rightarrow (*)^-_{<\kappa}$.

Similarly to WRP, for any regular $\kappa \geq \aleph_2$, the consistency of $(*)^+_{<\kappa^+}$ (denote by $(*)^+_{\leq\kappa}$) can be obtained by Levy collapsing a supercompact cardinal bigger than κ to become κ^+.

Let $\kappa \geq \aleph_2$ be a regular cardinal. Define:

$(*)^+_{<\kappa}$

Given any $\eta > \kappa$, for any structure $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \ldots \rangle$ in countable signature and any family $S = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ satisfying:

0. $[N]^{\aleph_0} \subseteq N$;
1. $\mathfrak{A} \restriction_N \prec \mathfrak{A}$;
2. for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is an stationary subset of $[N]^{\aleph_0}$.

Clearly $(*)^+_{<\kappa} \Rightarrow (*)_{<\kappa} \Rightarrow (*)^-_{<\kappa}$.
Similarly to WRP, for any regular $\kappa \geq \aleph_2$, the consistency of $(*)^+_{<\kappa^+}$ (denote by $(*)^+_{\leq\kappa}$) can be obtained by Levy collapsing a supercompact cardinal bigger than κ to become κ^+.

Diagonal Reflection Principle

$(*)_{<\kappa}$ is a variation of the following principle introduced by Sean Cox.

Let $\mathcal{C}$ be a class of sets of cardinality $\aleph_1$ and $\theta > \aleph_1$ be a cardinal of uncountable cofinality.

DRP$(\theta, \mathcal{C})$

There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ such that

1. $M \cap \mathcal{H}(\theta) \in \mathcal{C}$;
2. for all $R \in M$, if R is a stationary subset of $[\theta]^{\aleph_0}$, then $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

Lemma

$(*)_{<\aleph_2} \Leftrightarrow$ DRP(θ, IC_{ω_1}) holds for all regular $\theta \geq \aleph_2$

where IC_{ω_1} is the class of all internally club sets of size $\aleph_1$.

Diagonal Reflection Principle

$(*)_{<\kappa}$ is a variation of the following principle introduced by Sean Cox.

Let $\mathcal{C}$ be a class of sets of cardinality $\aleph_1$ and $\theta > \aleph_1$ be a cardinal of uncountable cofinality.

DRP$(\theta, \mathcal{C})$

There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ such that

1. $M \cap \mathcal{H}(\theta) \in \mathcal{C}$;
2. for all $R \in M$, if R is a stationary subset of $[\theta]^{\aleph_0}$, then $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

Lemma

$(*)_{<\aleph_2} \Leftrightarrow DRP(\theta, IC_{\omega_1})$ holds for all regular $\theta \geq \aleph_2$

where $\mathrm{IC}_{\omega 1}$ is the class of all internally club sets of size $\aleph_1$.

For $\kappa \geq \aleph_2$ we have:

Proposition

$$SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \Leftrightarrow (*)_{<\kappa}$$

Proposition

$$SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa) \Leftrightarrow (*)^{+}_{<\kappa}$$

In particular, for the case where $\kappa = \aleph_2$, we have

$$\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \aleph_2) \Leftrightarrow (*)_{<\aleph_2} + \mathrm{CH}$$

For $\kappa \geq \aleph_2$ we have:

Proposition

$$SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \Leftrightarrow (*)_{<\kappa}$$

Proposition

$$SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa) \Leftrightarrow (*)^{+}_{<\kappa}$$

In particular, for the case where $\kappa = \aleph_2$, we have

$$\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \aleph_2) \Leftrightarrow (*)_{<\aleph_2} + \mathrm{CH}$$

For $\kappa \geq \aleph_2$ we have:

Proposition

$$SDLS^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \Leftrightarrow (*)_{<\kappa}$$

Proposition

$$SDLS(\mathcal{L}^{\aleph_0, II}_{stat}, < \kappa) \Leftrightarrow (*)^{+}_{<\kappa}$$

In particular, for the case where $\kappa = \aleph_2$, we have

$$\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < \aleph_2) \Leftrightarrow (*)_{<\aleph_2} + \mathrm{CH}$$

1. Stationary Logic and reflection principles
2. Consistency Result

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{“}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{”}.$$

Proof.

- Start assuming $V \models$ “ MM $\wedge\, \exists\lambda$ supercompact cardinal”;
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models$ “$(*)^+_{<\aleph_3}$”;
- $\mathrm{Col}(\omega_2, <\lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models$ “$(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2$”.

□

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{“}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{”}.$$

Proof.

- Start assuming $V \models$ “MM $\wedge \exists\lambda$ supercompact cardinal”;
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models \text{“}(*)^+_{<\aleph_3}\text{”}$;
- $\mathrm{Col}(\omega_2, <\lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models \text{“}(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2\text{”}$.

□

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\text{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{“}\,(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\,\text{”}.$$

Proof.

- Start assuming $V \models$ “ MM $\wedge \exists\lambda$ supercompact cardinal”;
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models \text{“}\,(*)^+_{<\aleph_3}\,\text{”}$;
- $\mathrm{Col}(\omega_2, < \lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models \text{“}\,(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2\,\text{”}$.

□

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{“}\,(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\,\text{”}.$$

Proof.

- Start assuming $V \models$ “ MM $\wedge\, \exists\lambda$ supercompact cardinal”;
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models$ “ $(*)^+_{<\aleph_3}$ ”;
- $\mathrm{Col}(\omega_2, <\lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models$ “ $(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2$ ”.

□

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{“} (*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}} \text{”}.$$

Proof.

- Start assuming $V \models$ “ MM $\wedge\, \exists \lambda$ supercompact cardinal”;
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models$ “ $(*)^+_{<\aleph_3}$ ”;
- $\mathrm{Col}(\omega_2, < \lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models$ “ $(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2$ ”.

□

Consistency Result

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^+_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^-_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{``}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{''}.$$

Proof.

- Start assuming $V \models$ "MM $\wedge\, \exists\lambda$ supercompact cardinal";
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^-_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models \text{``}(*)^+_{<\aleph_3}\text{''}$;
- $\mathrm{Col}(\omega_2, < \lambda)$ preserves $(*)^-_{<\aleph_2}$;
- then, indeed $W \models \text{``}(*)^-_{<\aleph_2} \wedge (*)^+_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2\text{''}$.

□

Consistency Result

Notice that $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{\mathrm{stat}}, < 2^{\aleph_0})$ is always false (thus $(*)^{+}_{<2^{\aleph_0}}$ is also false). However the weaker principle $(*)^{-}_{<2^{\aleph_0}}$ is consistent. Actually, we have a simple example of a model W such that

$$W \models \text{``}\, (*)^{+}_{\leq 2^{\aleph_0}} \wedge (*)^{-}_{<2^{\aleph_0}} \,\text{''}.$$

Proof.

- Start assuming $V \models$ " MM $\wedge\, \exists\lambda$ supercompact cardinal";
- MM implies $2^{\aleph_0} = \aleph_2$ and also implies $(*)^{-}_{<\aleph_2}$;
- define $W := V^{\mathrm{Col}(\omega_2, <\lambda)}$. By collapsing λ to become $\aleph_3$ we obtain $W \models$ " $(*)^{+}_{<\aleph_3}$ ";
- $\mathrm{Col}(\omega_2, < \lambda)$ preserves $(*)^{-}_{<\aleph_2}$;
- then, indeed $W \models$ " $(*)^{-}_{<\aleph_2} \wedge (*)^{+}_{<\aleph_3} \wedge 2^{\aleph_0} = \aleph_2$ ".

□

For this proof we needed $2^{\aleph_0} = \aleph_2$ and $(2^{\aleph_0})^+ = \aleph_3$. This rises the question: is $(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}$ consistent with $2^{\aleph_0}$ being arbitrarily big? Is it consistent with the continuum having some large cardinal property?

Yes

For this proof we needed $2^{\aleph_0} = \aleph_2$ and $(2^{\aleph_0})^+ = \aleph_3$. This rises the question: is $(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}$ consistent with $2^{\aleph_0}$ being arbitrarily big? Is it consistent with the continuum having some large cardinal property?

Yes

Sketch of the proof

Assume GCH and assume there exists $\kappa < \lambda$ supercompact cardinals. We construct W such that

$$W \models \text{“} (*)^{+}_{\leq 2^{\aleph_0}} \wedge (*)^{-}_{<2^{\aleph_0}} \wedge 2^{\aleph_0} \text{ carries a } \sigma\text{-saturated ideal”}$$

We construct W by first adding κ many reals, and then we collapse λ to κ^{+}. However just simply adding reals (say, with a Cohen forcing) does not work.

We need to add the reals in a way such that we can extend some elementary embeddings for κ and λ in some nice extension.

For this we shall construct a forcing iteration.

Sketch of the proof

Assume GCH and assume there exists $\kappa < \lambda$ supercompact cardinals. We construct W such that

$$W \models \text{“} (*)^{+}_{\leq 2^{\aleph_0}} \wedge (*)^{-}_{<2^{\aleph_0}} \wedge 2^{\aleph_0} \text{ carries a } \sigma\text{-saturated ideal”}$$

We construct W by first adding κ many reals, and then we collapse λ to κ^{+}. However just simply adding reals (say, with a Cohen forcing) does not work.

We need to add the reals in a way such that we can extend some elementary embeddings for κ and λ in some nice extension.

For this we shall construct a forcing iteration.

Sketch of the proof

Assume GCH and assume there exists $\kappa < \lambda$ supercompact cardinals. We construct W such that

$$W \models \text{“} (*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}} \wedge 2^{\aleph_0} \text{ carries a } \sigma\text{-saturated ideal”}$$

We construct W by first adding κ many reals, and then we collapse λ to κ^+. However just simply adding reals (say, with a Cohen forcing) does not work.
We need to add the reals in a way such that we can extend some elementary embeddings for κ and λ in some nice extension.
For this we shall construct a forcing iteration.

Sketch of the proof

Assume GCH and assume there exists $\kappa < \lambda$ supercompact cardinals. We construct W such that

$$W \models \text{“} (*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}} \wedge 2^{\aleph_0} \text{ carries a } \sigma\text{-saturated ideal”}$$

We construct W by first adding κ many reals, and then we collapse λ to κ^+. However just simply adding reals (say, with a Cohen forcing) does not work.
We need to add the reals in a way such that we can extend some elementary embeddings for κ and λ in some nice extension.
For this we shall construct a forcing iteration.

Let $f : \kappa \longrightarrow \kappa$ be a Laver function such that $\forall \xi < \kappa, f(\xi) > \xi$ and for every cardinal $\eta \geq \lambda$ there is an η-supercompact embedding $j : V \longrightarrow M$ for κ such that

$$j(f)(\kappa) = \lambda$$

Since κ is supercompact, the set

$$S := \{\alpha < \kappa \;:\; \alpha \text{ is a Mahlo cardinal and } \forall \beta < \alpha, f(\beta) < \alpha\}$$

is stationary in κ.

Let $f : \kappa \longrightarrow \kappa$ be a Laver function such that $\forall \xi < \kappa, f(\xi) > \xi$ and for every cardinal $\eta \geq \lambda$ there is an η-supercompact embedding $j : V \longrightarrow M$ for κ such that

$$j(f)(\kappa) = \lambda$$

Since κ is supercompact, the set

$$S := \{\alpha < \kappa \,:\, \alpha \text{ is a Mahlo cardinal and } \forall \beta < \alpha, f(\beta) < \alpha\}$$

is stationary in κ.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

Rough description of the iteration

We define a mixed support iteration $\overrightarrow{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$:
(this iteration is a modification of a construction by Krueger)

- at step $\alpha \in S$, we collapse (via usual Lévy collapse) all the cardinals between α and $f(\alpha)$;
- at every other step, we add a Cohen real;
- This iteration can be seen as having "two parts":
  1. the Levy collapse part is an Easton support iteration;
  2. and the adding Cohen Reals part is a finite support product.

We have

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\kappa \text{ is weakly Mahlo and } 2^{\aleph_0} = \kappa\text{”}$$

Furthermore, $\mathbb{P}_\kappa$ is designed to have the following property:

Lemma (Key lemma)

For any $\eta \geq \lambda$, there is an η-supercompact embedding $j : V \longrightarrow M$ with critical point κ such that in M we have:

1. $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda) \lessdot j(\mathbb{P}_\kappa)$;
2. $\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$ “$j(\mathbb{P}_\kappa)/\dot{G}_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$ *is proper*”.

In particular, $j(\vec{\mathbb{P}})$ is an iteration of length $j(\kappa)$ such that

$$j(\vec{\mathbb{P}})_{\kappa+1} = \mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda).$$

Features of $\mathbb{P}_\kappa$

We have

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\, \kappa \text{ is weakly Mahlo and } 2^{\aleph_0} = \kappa \text{”}$$

Furthermore, $\mathbb{P}_\kappa$ is designed to have the following property:

Lemma (Key lemma)

For any $\eta \geq \lambda$, there is an η-supercompact embedding $j : V \longrightarrow M$ with critical point κ such that in M we have:

1. $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda) \lessdot j(\mathbb{P}_\kappa)$;
2. $\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)} \text{“}\, j(\mathbb{P}_\kappa)/\dot{G}_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$ *is proper”.*

In particular, $j(\overrightarrow{\mathbb{P}})$ is an iteration of length $j(\kappa)$ such that

$$j(\overrightarrow{\mathbb{P}})_{\kappa+1} = \mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda).$$

We have

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\,\kappa \text{ is weakly Mahlo and } 2^{\aleph_0} = \kappa\,\text{”}$$

Furthermore, $\mathbb{P}_\kappa$ is designed to have the following property:

Lemma (Key lemma)

For any $\eta \geq \lambda$, there is an η-supercompact embedding $j : V \longrightarrow M$ with critical point κ such that in M we have:

1. $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda) \lessdot j(\mathbb{P}_\kappa)$;
2. $\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$ “$j(\mathbb{P}_\kappa)/\dot{G}_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$ *is proper*”.

In particular, $j(\overrightarrow{\mathbb{P}})$ is an iteration of length $j(\kappa)$ such that

$$j(\overrightarrow{\mathbb{P}})_{\kappa+1} = \mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda).$$

Sketch of the proof

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa, <\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{“}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{”}$$

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{"}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{"}$$

Sketch of the proof

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa, <\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{“}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{”}$$

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{"}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{"}$$

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{“}(*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}}\text{”}$$

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{“} (*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}} \text{”}$$

- Our final model shall be $W := V^{\mathbb{P}_\kappa * \mathrm{Col}(\kappa, <\lambda)}$.
- In W, we have $2^{\aleph_0} = \kappa$ and $(2^{\aleph_0})^+ = \lambda$.
- Since $\mathbb{P}_\kappa$ is small, λ remains supercompact in $V^{\mathbb{P}_\kappa}$.
- Like we argued before, by collapsing λ to become κ^+ we get $W \models$ "$(*)^+_{<\lambda}$".
- By the key lemma, we can extend some supercompact embedding at κ in some proper extension of W.
- It follows that $(*)^-_{<\kappa}$ holds in W.

Therefore indeed

$$W \models \text{“} (*)^+_{\leq 2^{\aleph_0}} \wedge (*)^-_{<2^{\aleph_0}} \text{”}$$

We also sketch the following proof:

Lemma

Assume $V \models$ " GCH". Then:

$$\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)} \text{“}\kappa = 2^{\aleph_0} \textit{ carries a } \sigma\textit{-saturated ideal”}$$

Let $j : V \longrightarrow M$ be a λ-supercompact embedding, $\mathrm{crit}(j) = \kappa$ such that $j(f)(\kappa) = \lambda$, like before.
Let G be a $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda)$-generic over V.

Lemma

In $V[G]$, $j(\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda))/G$ is a projection of $\mathbb{R} \times \mathbb{S}$, where $\mathbb{S}$ is ccc and $\mathbb{R}$ is $< \lambda^+$-closed (in $M[G]$).

We also sketch the following proof:

Lemma

Assume $V \models$ " GCH". Then:

$$\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)} \text{``} \kappa = 2^{\aleph_0} \textit{ carries a } \sigma\textit{-saturated ideal''}$$

Let $j : V \longrightarrow M$ be a λ-supercompact embedding, $\mathrm{crit}(j) = \kappa$ such that $j(f)(\kappa) = \lambda$, like before.
Let G be a $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda)$-generic over V.

Lemma

In $V[G]$, $j(\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda))/G$ is a projection of $\mathbb{R} \times \mathbb{S}$, where $\mathbb{S}$ is ccc and $\mathbb{R}$ is $< \lambda^+$-closed (in $M[G]$).

We also sketch the following proof:

Lemma

Assume $V \models$ " GCH". Then:

$$\Vdash_{\mathbb{P}_\kappa * \mathrm{Col}(\kappa,<\lambda)} \text{``}\kappa = 2^{\aleph_0} \textit{ carries a } \sigma\textit{-saturated ideal''}$$

Let $j : V \longrightarrow M$ be a λ-supercompact embedding, $\mathrm{crit}(j) = \kappa$ such that $j(f)(\kappa) = \lambda$, like before.
Let G be a $\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda)$-generic over V.

Lemma

In $V[G]$, $j(\mathbb{P}_\kappa * \mathrm{Col}(\kappa, < \lambda))/G$ is a projection of $\mathbb{R} \times \mathbb{S}$, where $\mathbb{S}$ is ccc and $\mathbb{R}$ is $< \lambda^+$-closed (in $M[G]$).

Let $H_{\mathbb{S}}$ be a $\mathbb{S}$-generic over $V[G]$. It follows from the $< \lambda^+$-closedness of $\mathbb{R}$ that

Lemma

There is $H \in V[G * H_{\mathbb{S}}]$ such that j can be extended into an elementary embedding $J : V[G] \longrightarrow M[G * H]$.

Therefore, in $V[G]$, we can prove define

$$I := \{x \subseteq \kappa \ : \ \Vdash_{\mathbb{S}} \text{``} \kappa \notin J(x) \text{''}\}$$

Since $\mathbb{S}$ is ccc, it follows that I is a σ-saturated ideal. $\square$

Let $H_{\mathbb{S}}$ be a $\mathbb{S}$-generic over $V[G]$. It follows from the $< \lambda^+$-closedness of $\mathbb{R}$ that

Lemma

There is $H \in V[G * H_{\mathbb{S}}]$ such that j can be extended into an elementary embedding $J : V[G] \longrightarrow M[G * H]$.

Therefore, in $V[G]$, we can prove define

$$I := \{x \subseteq \kappa : \Vdash_{\mathbb{S}} \text{“} \kappa \notin J(x) \text{”}\}$$

Since $\mathbb{S}$ is ccc, it follows that I is a σ-saturated ideal. $\square$

Let $H_{\mathbb{S}}$ be a $\mathbb{S}$-generic over $V[G]$. It follows from the $< \lambda^+$-closedness of $\mathbb{R}$ that

Lemma

There is $H \in V[G * H_{\mathbb{S}}]$ such that j can be extended into an elementary embedding $J : V[G] \longrightarrow M[G * H]$.

Therefore, in $V[G]$, we can prove define

$$I := \{x \subseteq \kappa \ : \ \Vdash_{\mathbb{S}} \text{``} \kappa \notin J(x) \text{''}\}$$

Since $\mathbb{S}$ is ccc, it follows that I is a σ-saturated ideal. $\square$

Let $H_{\mathbb{S}}$ be a $\mathbb{S}$-generic over $V[G]$. It follows from the $< \lambda^+$-closedness of $\mathbb{R}$ that

Lemma

There is $H \in V[G * H_{\mathbb{S}}]$ such that j can be extended into an elementary embedding $J : V[G] \longrightarrow M[G * H]$.

Therefore, in $V[G]$, we can prove define

$$I := \{x \subseteq \kappa \ : \ \Vdash_{\mathbb{S}} \text{“}\kappa \notin J(x)\text{”}\}$$

Since $\mathbb{S}$ is ccc, it follows that I is a σ-saturated ideal. $\square$

Thank you very much!

- Sean Cox.
  The diagonal reflection principle.
  Proc. Amer. Math. Soc., 140(8):2893–2902, 2012.
- Bernhard König.
  Generic compactness reformulated.
  Arch. Math. Logic, 43(3):311–326, 2004.
- John Krueger.
  A general Mitchell style iteration.
  MLQ Math. Log. Q., 54(6):641–651, 2008.