Global Chang's Conjecture and singular cardinals

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Introduction

Theorem (Lowenheim-Skolem)

Let $\mathfrak{A}$ be an infinite model in a countable first-order language. For every infinite cardinal $\kappa \leq |\mathfrak{A}|$, there is an elementary $\mathfrak{B} \prec \mathfrak{A}$ of size κ.

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If $\kappa_1 = \kappa_0^+$ and $\mu_1 = \mu_0^+$, this is equivalent to an analogue of Lowenheim-Skolem for a logic between first and second order. This logic includes a quantifier Qx, where $Qx\varphi(x)$ is valid when the number of x's satisfying $\varphi(x)$ is equal to the size of the model.

GCH restrictions

Lemma

Suppose $\kappa, \lambda \leq \delta$ *and* $\kappa^{\lambda} \geq \delta$. *Then there is a structure* $\mathfrak{A}$ *on* δ *such that for every* $\mathfrak{B} \prec \mathfrak{A}$,

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Corollary

If $(\kappa_1, \kappa_0) \twoheadrightarrow (\mu_1, \mu_0)$, $\nu \leq \kappa_0$, *and* $\kappa_0^\nu \geq \kappa_1$, *then* $\mu_0^{\min(\mu_0, \nu)} \geq \mu_1$.

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Global Chang's Conjecture

For all infinite cardinals $\mu < \kappa$ with $\mathrm{cf}(\mu) \leq \mathrm{cf}(\kappa)$, $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$.

Approximations to GCC

Theorem (E.-Hayut)

It is consistent relative to a huge cardinal that $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$ *holds whenever* $\omega \leq \mu < \kappa$ *and* κ *is regular.*

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It is consistent relative to a huge cardinal that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *while for all* $n < m < \omega$, $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$.

It turns out that this was optimal; it is the longest initial segment of cardinals on which GCC can hold.

Some useful lemmas

We say $(\kappa_1, \kappa_0) \twoheadrightarrow_\nu (\mu_1, \mu_0)$ holds when for all $\mathfrak{A}$ on κ_1, there is $\mathfrak{B} \prec \mathfrak{A}$ of size μ_1 with $|\mathfrak{B} \cap \kappa_0| = \mu_0$, and $\nu \subseteq \mathfrak{B}$. This is preserved under ν^+-c.c. forcing.

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1. *If* $\kappa_0 = \mu_0^{+\nu}$, *then* $(\kappa_1, \kappa_0) \twoheadrightarrow_{\mu_0} (\mu_1, \mu_0)$.
2. *If* $\lambda \leq \mu_0$ *and there is* $\kappa \leq \kappa_0$ *such that* $\kappa_0 = \kappa^{+\nu}$ *and* $\kappa^\lambda \leq \kappa_0$, *then* $(\kappa_1, \kappa_0) \twoheadrightarrow_\lambda (\mu_1, \mu_0)$.

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Lemma

Suppose $\mu^{<\nu} = \mu$, *and* $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$. *Then* $(\kappa^+, \kappa) \twoheadrightarrow_\nu (\mu^+, \mu)$.

Scales

If κ is a singular cardinal, and $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ is an increasing sequence of regular carindals cofinal in κ, $\langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ is a *scale* for κ if it is increasing and dominating in the product (mod bounded). Shelah proved that singular κ always carry scales of length κ^+.

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A scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is *good at* α when there is a pointwise increasing sequence $\langle g_i : i < \mathrm{cf}(\alpha) \rangle$ such that this sequence and $\langle f_\beta : \beta < \alpha \rangle$ are cofinal in each other. A scale is *bad at* α when it is not good at α.

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Lemma (Folklore)

If κ is singular and $(\kappa^+, \kappa) \twoheadrightarrow_{\mathrm{cf}(\kappa)} (\mu^+, \mu)$ and $\mu \geq \mathrm{cf}(\kappa)$, then there is no good scale for κ. Moreover, every scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ for κ is bad at stationarily many α of cofinality μ^+.

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We use a few known results. First due to Shelah: If $\lambda < \kappa$ are regular, $S^{\kappa^+}_\lambda$ is the union of κ sets each carrying a partial square.

Corollary

If κ is regular, then there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ forming a "partial weak square." $\mathcal{C}_\alpha = \emptyset$ only when $\mathrm{cf}(\alpha) = \kappa$.

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Lemma (Foreman-Magidor)

Suppose κ *is regular and* $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$*. There is a structure* $\mathfrak{A}$ *on* κ^+ *such that any* $\mathfrak{B} \prec \mathfrak{A}$ *witnessing CC has* $\mathrm{cf}(\mathfrak{B} \cap \kappa) = \mathrm{cf}(\mu)$*.*

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This is enough to carry out the well-known construction of a good scale from weak sqaure.

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Theorem (E.-Hayut)

Suppose there is a supercompact and $\alpha < \beta$ are countable limit ordinals. Then there is a forcing extension in which $(\aleph_{\beta+1}, \aleph_\beta) \twoheadrightarrow (\aleph_{\alpha+1}, \aleph_\alpha)$.

CC between singulars with the same tail type

Lemma

Let $\eta < \kappa$ *be such that* $\kappa^{+\eta}$ *is a strong limit cardinal and* κ *is* $\kappa^{+\eta+1}$*-supercompact. There is* $\mu < \kappa$ *such that* $\mathrm{Col}(\mu^{+\eta+2}, \kappa)$ *forces* $(\kappa^{+\eta+1}, \kappa^{+\eta}) \twoheadrightarrow_{\mu^{+\eta}} (\mu^{+\eta+1}, \mu^{+\eta})$.

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Proof sketch: If not, then for every $\alpha < \kappa$ there is a $\mathrm{Col}(\alpha^{+\eta+2}, \kappa)$-name for a function $\dot{F}_\alpha : (\alpha^{+\eta+1})^{<\omega} \to \alpha^{+\eta}$ witnessing failure.

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Let $\dot{F} = j(\langle \dot{F}_\alpha \rangle)(\kappa)$ and $X = j[\kappa^{+\eta+1}]$. There is a sequence $\langle x_\beta : \beta < \kappa^{+\eta+1} \rangle \subseteq X^{<\omega}$ such that some condition p forces $\langle \dot{F}(j(x_\beta)) : \beta < \kappa^{+\eta+1} \rangle$ is an increasing sequence below $j(\kappa)^{+\eta}$.

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By elementarity, for each pair $\beta < \gamma$, there is $\alpha < \kappa$ and $p \in \mathrm{Col}(\alpha^{+\eta+2}, \kappa)$ such that $p \Vdash \dot{F}_\alpha(x_\beta) < \dot{F}_\alpha(x_\gamma) < \kappa^{+\eta}$.

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Applying Erdos-Rado and that $\kappa^{+\eta}$ is a singular strong limit, there are fixed α, δ, p such that for all $\beta < \gamma < \kappa^{+\delta+1}$, $p \Vdash \dot{F}_\alpha(x_\beta) < \dot{F}_\alpha(x_\gamma) < \kappa^{+\delta}$.

Singular GCC up to $\aleph_{\omega^2}$ using Gitik-Sharon forcing

We construct a forcing $\mathbb{P}$ as follows. Suppose $\mu < \kappa$ is regular. For each $n \geq 1$, let U_n be a normal measure on $\mathcal{P}_\kappa(\kappa^{+n})$, and let $j_n : V \to M_n \cong \mathrm{Ult}(V, U_n)$ be the ultrapower embedding. By the closure of the ultrapowers and GCH, we may choose an M_n-generic $G_n \subseteq \mathrm{Col}(\kappa^{\omega+2}, j_n(\kappa))^{M_n}$.

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3. For $i \geq n$, dom $F_i \in U_i$, and for $x \in$ dom F_i, $\kappa_x := x \cap \kappa$ is a cardinal greater than $|x_{n-1}| + \sup(\mathrm{ran}\, f_{n-1})$.
4. For $i \geq n$ and $x \in$ dom F_i, $F_i(x) \in \mathrm{Col}(\kappa_x^{+\omega+2}, \kappa)$, and $[F_i]_{U_i} \in G_i$.

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Let p be a condition of length $\ell \geq 1$ forcing $\kappa_i \in A$ for all i. Let $q \leq^* p$ decide the statement $\sigma :=$ "$(\kappa^+, \kappa) \twoheadrightarrow (\kappa_\ell^{+\omega+1}, \kappa_\ell^{+\omega})$." We claim that $q \Vdash \sigma$.

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We define an iteration of ultrapowers. Let $N_\ell = V$. Given a commuting system of elementary embeddings $j_{m,m'} : N_m \to N_{m'}$ for $\ell \leq m \leq m' \leq n$, let $j_{n,n+1} : N_n \to \mathrm{Ult}(N_n, j_{\ell,n}(U_{n+1})) = N_{n+1}$ be the ultrapower embedding, and let $j_{m,n+1} = j_{n,n+1} \circ j_{m,n}$ for $m < n$. For each $n \geq \ell$, let $j_{n,\omega} : N_n \to N_\omega$ be the direct limit embedding. N_ω is well-founded.

Singular GCC up to $\aleph_{\omega^2}$ using Gitik-Sharon forcing

By GCH and some counting arguments, $j_{0,\omega}(\kappa) = \kappa^{+\omega}$ and $j_{0,\omega}(\kappa^{+\omega+1}) = \kappa^{+\omega+1}$.

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Let $\text{stem}(p_1) = \langle f_0, x_1, f_1, ..., x_\ell, f_\ell \rangle$, and let $C_0 \times ... \times C_\ell \subseteq \text{Col}(\mu, \kappa_1) \times \text{Col}(\kappa_1^{+\omega+2}, \kappa_2) \times ... \times \text{Col}(\kappa_\ell^{+\omega+2}, \kappa)$ be generic over V containing $\langle f_0, ..., f_\ell \rangle$.

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Claim 1: $\langle C_0, x_1, C_1, ..., x_\ell, C_\ell, x_{\ell+1}, C_{\ell+1}, \ldots \rangle$ generates a generic for $j_{\ell,\omega}(\mathbb{P})$ over N_ω containing $j_{\ell,\omega}(q)$.

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Claim 2: Let G be the generated filter for $j_{\ell,\omega}(\mathbb{P})$. Then $N_\omega[G]$ is closed under κ-sequences from $V[C_0]...[C_\ell]$.

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By the Lemma, $V[C_0]...[C_\ell] \models (\kappa^{+\omega+1}, \kappa^{+\omega}) \twoheadrightarrow (\kappa_\ell^{+\omega+1}, \kappa_\ell^{+\omega})$.

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To get CCs down to $(\aleph_1, \aleph_0)$, collapse $\kappa_1^{+\omega}$ to ω.

Singular GCC below $\aleph_{\omega^\omega}$

Lemma

Suppose GCH and δ is $\delta^{+\omega+1}$-supercompact and Vopenka. Then in a forcing extension, V_δ satisfies

1. *ZFC + GCH*
2. *There is a supercompact.*
3. *For all $\beta > \alpha$, $(\beta^{+\omega+1}, \beta^{+\omega}) \twoheadrightarrow (\alpha^{+\omega+1}, \alpha^{+\omega})$, and this is indestructible any $(\alpha^{+\omega+1}, \alpha^{+\omega+1})$-distributive forcing of size $< \beta^{+\omega}$.*

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Starting from a model as above, we introduce a Radinized version of Gitik-Sharon forcing, which adds a club of ordertype ω^ω of former large cardinals, using a $(+\omega^2)$-supercompactness measure. We go as far as we can with "converting ordinal addition into ordinal multiplication."

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In the general case we work with sequences of ultrafilters paired with collapse filters $\langle U_\alpha, K_\alpha : \alpha < \omega \cdot n \rangle$ such that:

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Suppose $n > 1$, we have defined GS_m for $m < n$. A pair (μ, d), where μ is a regular cardinal and d is an appropriate sequence of filters of length $\omega \cdot m$, determines a forcing $\mathbb{P}(\mu, d) \in \mathrm{GS}_m$.

Elements of $\mathbb{P}(\mu, d) \in \mathrm{GS}_n$

Conditions take the form: $p = \langle f_0, e_1, (x_1, a_1), f_1, \ldots, e_\ell, (x_\ell, a_\ell), f_\ell, \vec{F} \rangle$.

1. For $1 \leq i \leq \ell$:
   1. $|x_i| < \kappa$, $x_i \prec H_{\kappa^{+\omega\cdot(n-1)+i+1}}$, $\kappa_i := x_i \cap \kappa$ is inaccessible, the transtive collapse of x_i is $H_{\kappa_i^{+\omega\cdot(n-1)+i+1}}$, and $\langle U_\alpha, K_\alpha : \alpha < \omega\cdot(n-1) \rangle \in x_i$.

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Singular GCC

Finally, we define GS_ω by diagonally weaving together the GS_n. For the Prikry property, we prove inductively a "decisive coloring lemma." We do a similar argument for CC between singulars of different tail types by iterating ultrapowers up to ω^ω times.

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Theorem (E.-Hayut)

Suppose there is a supercompact κ with κ^+ many measurables above, and ξ is a countable ordinal. Then there is a forcing extension in which for all $\alpha < \xi$, $(\aleph_{\alpha+1}, \aleph_\alpha) \twoheadrightarrow (\aleph_1, \aleph_0)$.

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Thank you for your attention!