On preserving AD by forcing

Daisuke Ikegami
(Shibaura Institute of Technology)

Joint work with Nam Trang

From now on...

We work in ZF+$\mathsf{DC}_{\mathbb{R}}$.

The choice priciple $\mathsf{DC}_{\mathbb{R}}$ states the following:

For any $A \subseteq \mathbb{R} \times \mathbb{R}$, if $(\forall x \in \mathbb{R})\ (\exists x \in \mathbb{R})\ (x, y) \in A$,

then $(\exists f : \omega \to \mathbb{R})\ (\forall n \in \omega)\ \big(f(n), f(n+1)\big) \in A$.

Main Open Question

Question

Assume AD. Can one find a poset P which adds a new real while preserving AD?

The Axiom of Determinacy (AD) states that one of the players has a winning strategy for any Gale-Stewart game with a payoff set as a subset of the Baire space ω^ω.

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Example

Cohen forcing destroys AD.

Reason: The reals in the ground model does NOT have the Baire property in any Cohen forcing extension.

Main Result

Theorem (I., Trang)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".
If a poset P increases Θ, then P destroys AD.

Definition

- A set of reals is ∞-Borel if it is λ-Borel for some λ.
- The ordinal Θ is defined as follows:

$$\Theta = \sup\{\gamma \mid \gamma \text{ is a surjective image of } \mathbb{R}\}$$

Three remarks on the main result

Remark

1. The assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential, i.e., one can find a counter example of the theorem without this assumption.
2. The assumption "Every set of reals is ∞-Borel" is non-trivial. In fact, under this assumption, AD implies that every set of reals is Ramsey. On the other hand, it is still open whether AD itself implies that every set of reals is Ramsey.
3. In ZFC, $\Theta = (2^{\aleph_0})^+$, but under AD, Θ is quite large; it is a limit of measurable cardinals.

Background

Kunen: There is NO non-trivial & elementary $j\colon V \to V$ such that $(V, \in, j) \models$ ZFC.

Open Question: How about replacing ZFC above with ZF only?

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Hamkins et.al.: There is NO non-trivial & elementary $j \colon V \to V[G]$ such that $(V[G], \in, j) \models$ ZFC, where $V[G]$ is a set generic extension of V.

Woodin???: It is consistent to have $j \colon V \to V[G]$ as above if one demands $(V[G], \in, j) \models$ ZF only.
But in Woodin's example, $j \restriction \mathrm{Ord} = \mathrm{id}$.

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Question

How about demanding $(V[G], \in, j) \models \mathrm{ZF}+\mathrm{AD}$ and $V[G]$ has more reals than V?

Background ctd.

Theorem (Woodin)

Assume AD. If M is an inner model of ZF and V has more reals than M, then ω_1^M must be countable.

Corollary

If $j\colon V \to V[G]$ is non-trivial & elementary, and $V[G]$ is a model of AD and has more reals than V, then $\mathrm{crit}(j) = \omega_1^V$.

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Coming back to the first question:

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Assume AD. Can one find a poset P which adds a new real while preserving AD?

The main result stated again

Theorem (I., Trang)

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If a poset P increases Θ, then P destroys AD.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then $\mathrm{HOD} = \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

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Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^V$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

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But V is definable in a set generic extension of HOD.

So $X^\# \in \mathrm{HOD} = \mathrm{L}[X]$, contradiction!

Coming back to the 1st remark

Remark (stated again)

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Fact (Solovay???)

It is consistent that AD holds, every set of reals is ∞-Borel, and there is a set of reals A which is NOT in the model $\mathrm{HOD}(\mathbb{R})$.

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Example

Setting $M = \mathrm{HOD}(\mathbb{R})$ with $A \notin \mathrm{HOD}(\mathbb{R})$, M satisfies the assumptions of the Main Theorem except $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$, and there is a set generic extension $M[G]$ of M such that $M[G]$ satisfies AD while $\Theta^{M[G]} > \Theta^{M}$.

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The P for $M[G]$ is a Vopenka-like forcing with $A \in M[G] \subseteq V$ such that $\mathbb{R}^M = \mathbb{R}^{M[G]}$. This is enough to guarantee the desired properties of $M[G]$.

Further results by other researchers

Theorem (Chan, Jackson)

Assume AD. Then

1. any non-trivial well-orderable forcing of length less than Θ destroys AD, and
2. if Θ is regular, then any non-trivial forcing which is a surjective image of $\mathbb{R}$ destroys AD.

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1. any non-trivial well-orderable forcing of length less than Θ destroys AD, and
2. if Θ is regular, then any non-trivial forcing which is a surjective image of $\mathbb{R}$ destroys AD.

The demanded properties of P to preserve AD while adding new reals if $V = \mathrm{L}\big(\mathcal{P}(\mathbb{R})\big)$:

1. P needs to collapse ω_1 while preserving Θ,
2. for any cardinal $\kappa < \Theta$ in V^P, NO club subset of κ in V witnesses the strong partition property of κ in V^P while there are unboundedly many strong parition property cardinals in Θ in V^P.

THE END.