Winter School 2019

January 26th–February 2nd 2019
Hejnice, Czech Republic

Invited speakers

- James Cummings
- Miroslav Hušek
- Wiesław Kubiś
- Jordi Lopez-Abad

www.winterschool.eu

Logic Colloquium 2019

August 11th–16th 2019, Prague, Czech Republic

www.lc2019.cz

Program Committee

- Andrew Arana
- Lev Beklemishev (chair)
- Agata Ciabattoni
- Russell Miller
- Martin Otto
- Pavel Pudlák
- Stevo Todorčević
- Alex Wilkie

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

David Chodounský

Institute of Mathematics CAS

joint work with Vera Fischer and Jan Grebík

Free sequences in topological spaces

A sequence of points $\langle x_\alpha \mid \alpha < \gamma \rangle$ in a topological space is a free sequence if the topological closure of $\langle x_\alpha \mid \alpha < \beta \rangle$ is disjoint from the topological closure of $\langle x_\alpha \mid \beta \leq \alpha < \gamma \rangle$ for each $\beta \leq \gamma$.

Free sequences in topological spaces

A sequence of points $\langle x_\alpha \mid \alpha < \gamma \rangle$ in a topological space is a free sequence if the topological closure of $\langle x_\alpha \mid \alpha < \beta \rangle$ is disjoint from the topological closure of $\langle x_\alpha \mid \beta \leq \alpha < \gamma \rangle$ for each $\beta \leq \gamma$.

- Introduced by Alexander Arhangel'skiĭ.
- Related to the tightness of the space of the space.
- Used to prove $|X| \leq 2^{\chi(X)L(X)}$.

Free sequences in Boolean algebras

For $a \in \mathbf{B}$ denote $\quad a^0 = \mathbf{1} - a \quad$ and $\quad a^1 = a$.

Free sequences in Boolean algebras

For $a \in \mathbf{B}$ denote $a^0 = \mathbf{1} - a$ and $a^1 = a$.

Definition (Don Monk)

Sequence $A = \langle a_\alpha \mid \alpha \in \gamma \rangle$ of elements of $\mathbf{B}$ of ordinal length γ is a *free sequence* if the family $\mathcal{C}_\beta = \{ a_\alpha^1 \mid \alpha < \beta \} \cup \{ a_\alpha^0 \mid \beta \leq \alpha < \gamma \}$ is centered for each $\beta \leq \gamma$.

Free sequences in Boolean algebras

For $a \in \mathbf{B}$ denote $\quad a^0 = \mathbf{1} - a \quad$ and $\quad a^1 = a.$

Definition (Don Monk)

Sequence $A = \langle\, a_\alpha \mid \alpha \in \gamma \,\rangle$ of elements of $\mathbf{B}$ of ordinal length γ is a *free sequence* if the family $\mathcal{C}_\beta = \{\, a^1_\alpha \mid \alpha < \beta \,\} \cup \{\, a^0_\alpha \mid \beta \leq \alpha < \gamma \,\}$ is centered for each $\beta \leq \gamma$.

A free sequence is *maximal* if it is maximal with respect to end-extension.

Free sequences in Boolean algebras

For $a \in \mathbf{B}$ denote $\quad a^0 = \mathbf{1} - a \quad$ and $\quad a^1 = a.$

Definition (Don Monk)

Sequence $A = \langle\, a_\alpha \mid \alpha \in \gamma \,\rangle$ of elements of $\mathbf{B}$ of ordinal length γ is a *free sequence* if the family $\mathcal{C}_\beta = \{\, a^1_\alpha \mid \alpha < \beta \,\} \cup \{\, a^0_\alpha \mid \beta \leq \alpha < \gamma \,\}$ is centered for each $\beta \leq \gamma$.

A free sequence is *maximal* if it is maximal with respect to end-extension.
Monk studied the cardinal spectrum of cardinalities of maximal free sequences for a given $\mathbf{B}$.

$$\mathfrak{f}(\mathbf{B}) = \min\{\, |A| \mid A \text{ is a maximal free sequence in } \mathbf{B} \,\}$$

Question (Monk)

What is the relation of $\mathfrak{f}(\mathbf{B})$ *and* $\mathfrak{u}(\mathbf{B})$*?*

Free sequences in Boolean algebras

Definition (Don Monk)

Sequence $A = \langle a_\alpha \mid \alpha \in \gamma \rangle$ of elements of $\mathbf{B}$ of ordinal length γ is a *free sequence* if the family $\mathcal{C}_\beta = \{ a_\alpha^1 \mid \alpha < \beta \} \cup \{ a_\alpha^0 \mid \beta \leq \alpha < \gamma \}$ is centered for each $\beta \leq \gamma$.

A free sequence is *maximal* if it is maximal with respect to end-extension.
Monk studied the cardinal spectrum of cardinalities of maximal free sequences for a given $\mathbf{B}$.

$$\mathfrak{f}(\mathbf{B}) = \min\{ |A| \mid A \text{ is a maximal free sequence in } \mathbf{B} \}$$

Question (Monk)

What is the relation of $\mathfrak{f}(\mathbf{B})$ *and* $\mathfrak{u}(\mathbf{B})$*?*

Theorem (K. Selker)

It is consistent that there is $\mathbf{B}$ *such that* $\omega = \mathfrak{f}(\mathbf{B}) < \mathfrak{u}(\mathbf{B}) = \omega_1$ *(and* CH*).*

Free sequences in $\mathcal{P}(\omega)/\text{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\text{fin}$.
Denote

$$\text{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a^0_\alpha \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\mathrm{fin}$.
Denote

$$\mathrm{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Every strictly $\subset^*$ descending sequence is a free sequence.

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\mathrm{fin}$.
Denote

$$\mathrm{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Every strictly $\subset^*$ descending sequence is a free sequence.

If a free sequence A generates an ultrafilter, then A is maximal.

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\mathrm{fin}$.
Denote

$$\mathrm{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Every strictly $\subset^*$ descending sequence is a free sequence.

If a free sequence A generates an ultrafilter, then A is maximal.

Lemma

There is a free sequence which does not generate an ultrafilter.

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\mathrm{fin}$.
Denote

$$\mathrm{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Every strictly $\subset^*$ descending sequence is a free sequence.

If a free sequence A generates an ultrafilter, then A is maximal.

Lemma

There is a free sequence which does not generate an ultrafilter.

Assume $\omega = X \cup Y$, $A = \langle a_\alpha \subset X \mid \alpha \in \gamma \rangle$, $B = \langle b_\alpha \subset Y \mid \alpha \in \gamma \rangle$ are maximal free sequences.
For $\langle \alpha, i \rangle \in \gamma \times 2$ let $c_{\alpha,i} = a_\alpha \cup b_{\alpha+i}$.
$C = \langle c_{\alpha,i} \mid \langle \alpha, i \rangle \in \gamma \times 2 \rangle$ is a maximal free sequence.

Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$

Let A be a free sequence in $\mathcal{P}(\omega)/\mathrm{fin}$.
Denote

$$\mathrm{comb}(A) = \left\{ \bigcap_{\alpha\in\Gamma} a_\alpha \cap \bigcap_{\alpha\in\Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

Every strictly $\subset^*$ descending sequence is a free sequence.

If a free sequence A generates an ultrafilter, then A is maximal.

Lemma

There is a free sequence which does not generate an ultrafilter.

Assume $\omega = X \cup Y$, $A = \langle a_\alpha \subset X \mid \alpha \in \gamma \rangle$, $B = \langle b_\alpha \subset Y \mid \alpha \in \gamma \rangle$ are maximal free sequences.
For $\langle \alpha, i \rangle \in \gamma \times 2$ let $c_{\alpha,i} = a_\alpha \cup b_{\alpha+i}$.
$C = \langle c_{\alpha,i} \mid \langle \alpha, i \rangle \in \gamma \times 2 \rangle$ is a maximal free sequence.

Lemma

The length of a maximal free sequence may not be a limit ordinal.

Cardinality considerations

$\mathcal{X} \subset [\omega]^\omega$ is an independent system if for every function $f : \mathcal{X} \to 2$ is the family $\{ a^{f(a)} \mid a \in \mathcal{X} \}$ centered.

$$\mathfrak{i} = \min\{ |\mathcal{X}| \mid \mathcal{X} \text{ is a maximal independent system } \}$$

Cardinality considerations

$\mathcal{X} \subset [\omega]^\omega$ is an independent system if for every function $f : \mathcal{X} \to 2$ is the family $\{ a^{f(a)} \mid a \in \mathcal{X} \}$ centered.

$$\mathfrak{i} = \min\{ |\mathcal{X}| \mid \mathcal{X} \text{ is a maximal independent system } \}$$

$\mathcal{R} \subset [\omega]^\omega$ is a reaping family if for every $a \in [\omega]^\omega$ there is $r \in \mathcal{R}$ such that $r \subset^* a$ or $r \cap a =^* \emptyset$.

$$\mathfrak{r} = \min\{ |\mathcal{R}| \mid \mathcal{R} \text{ is a reaping family } \}$$

Cardinality considerations

$\mathcal{X} \subset [\omega]^\omega$ is an independent system if for every function $f : \mathcal{X} \to 2$ is the family $\{\, a^{f(a)} : a \in \mathcal{X} \,\}$ centered.

$$\mathfrak{i} = \min\{\, |\mathcal{X}| : \mathcal{X} \text{ is a maximal independent system } \}$$

$\mathcal{R} \subset [\omega]^\omega$ is a reaping family if for every $a \in [\omega]^\omega$ there is $r \in \mathcal{R}$ such that $r \subset^* a$ or $r \cap a =^* \emptyset$.

$$\mathfrak{r} = \min\{\, |\mathcal{R}| : \mathcal{R} \text{ is a reaping family } \}$$

$$\mathfrak{u} = \min\{\, |\mathcal{V}| : \mathcal{V} \text{ is an ultrafilter base } \}$$

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

The π-character $\pi\chi(\mathcal{U})$ is the minimal cardinality of a π-base of $\mathcal{U}$.

$\mathcal{B} \subset [\omega]^\omega$ is a π-base of $\mathcal{U}$ for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}$, $B \subset^* U$.

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

The π-character $\pi\chi(\mathcal{U})$ is the minimal cardinality of a π-base of $\mathcal{U}$.

$\mathcal{B} \subset [\omega]^\omega$ is a π-base of $\mathcal{U}$ for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}$, $B \subset^* U$.

$$\mathfrak{u}^* = \min\{\, \chi(\mathcal{U}) \mid \mathcal{U} \text{ is an ultrafilter such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) \,\}$$

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

The π-character $\pi\chi(\mathcal{U})$ is the minimal cardinality of a π-base of $\mathcal{U}$.

$\mathcal{B} \subset [\omega]^\omega$ is a π-base of $\mathcal{U}$ for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}$, $B \subset^* U$.

$$\mathfrak{u}^* = \min\{\, \chi(\mathcal{U}) \mid \mathcal{U} \text{ is an ultrafilter such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) \,\}$$

Question (Brendle–Shelah)

Does there exist an ultrafilter $\mathcal{U}$ such that $\chi(\mathcal{U}) = \pi\chi(\mathcal{U})$?

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

The π-character $\pi\chi(\mathcal{U})$ is the minimal cardinality of a π-base of $\mathcal{U}$.

$\mathcal{B} \subset [\omega]^\omega$ is a π-base of $\mathcal{U}$ for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}$, $B \subset^* U$.

$$\mathfrak{u}^* = \min\{\, \chi(\mathcal{U}) \mid \mathcal{U} \text{ is an ultrafilter such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) \,\}$$

Question (Brendle–Shelah)

Does there exist an ultrafilter $\mathcal{U}$ such that $\chi(\mathcal{U}) = \pi\chi(\mathcal{U})$?

Theorem (Bell–Kunen)

There is (in ZFC) an ultrafilter $\mathcal{U}$ such that $\pi\chi(\mathcal{U}) = \operatorname{cof} \mathfrak{c}$.

Let $\mathcal{U}$ be a non-principal ultrafilter.

The character $\chi(\mathcal{U})$ of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$.

The π-character $\pi\chi(\mathcal{U})$ is the minimal cardinality of a π-base of $\mathcal{U}$.

$\mathcal{B} \subset [\omega]^\omega$ is a π-base of $\mathcal{U}$ for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}$, $B \subset^* U$.

$$\mathfrak{u}^* = \min\{\, \chi(\mathcal{U}) \mid \mathcal{U} \text{ is an ultrafilter such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) \,\}$$

Question (Brendle–Shelah)

Does there exist an ultrafilter $\mathcal{U}$ such that $\chi(\mathcal{U}) = \pi\chi(\mathcal{U})$?

Theorem (Bell–Kunen)

There is (in ZFC) an ultrafilter $\mathcal{U}$ such that $\pi\chi(\mathcal{U}) = \operatorname{cof} \mathfrak{c}$.

Theorem (Balcar–Simon)

$\mathfrak{r} = \min\{\, \pi\chi(\mathcal{U}) \mid \mathcal{U}$ *is a non-principal ultrafilter* $\}$

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Lemma

$\mathfrak{r} \leq \mathfrak{f} \leq \mathfrak{u}^*$

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Lemma

$\mathfrak{r} \leq \mathfrak{f} \leq \mathfrak{u}^*$

If A is a maximal free sequence, then $\mathrm{comb}(A)$ *is a reaping family.*

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Lemma

$\mathfrak{r} \leq \mathfrak{f} \leq \mathfrak{u}^*$

Corollary

If $\mathfrak{r} = \mathfrak{u}$, *then* $\mathfrak{f} = \mathfrak{u} = \mathfrak{r}$.

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Lemma

$\mathfrak{r} \leq \mathfrak{f} \leq \mathfrak{u}^*$

Corollary

If $\mathfrak{r} = \mathfrak{u}$, *then* $\mathfrak{f} = \mathfrak{u} = \mathfrak{r}$.

Observation

Miller model: $\omega_1 = \mathfrak{u} = \mathfrak{f} < \mathfrak{i} = \mathfrak{c} = \omega_2$.

Fact

- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$
- If $\mathfrak{r} = \mathfrak{u}$, then $\mathfrak{u}^* = \mathfrak{u}$.

$\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\mathrm{fin})$

Lemma

$\mathfrak{r} \leq \mathfrak{f} \leq \mathfrak{u}^*$

Corollary

If $\mathfrak{r} = \mathfrak{u}$*, then* $\mathfrak{f} = \mathfrak{u} = \mathfrak{r}$.

Observation

Miller model: $\omega_1 = \mathfrak{u} = \mathfrak{f} < \mathfrak{i} = \mathfrak{c} = \omega_2$.

Question

Is $\mathfrak{i} < \mathfrak{f}$ *consistent with* ZFC*?*
What about $\mathfrak{r} < \mathfrak{f}$*?* $\mathfrak{u} < \mathfrak{f}$*?*

Theorem

$\mathfrak{i} = \mathfrak{f} < \mathfrak{u}$ *is consistent with* ZFC.

Theorem

$\mathfrak{i} = \mathfrak{f} < \mathfrak{u}$ *is consistent with* ZFC.

Denote $\mathbf{C}_\kappa = \{ h \colon \kappa \to 2 \mid |h| < \omega \}$ ordered by reverse inclusion.

Theorem

$\mathfrak{i} = \mathfrak{f} < \mathfrak{u}$ *is consistent with* ZFC.

Denote $\mathbf{C}_\kappa = \{ h \colon \kappa \to 2 \mid |h| < \omega \}$ ordered by reverse inclusion.
A forcing $\mathbf{P}$ is s *Cohen-preserving* if for each dense $D \subset \mathbf{C}_\kappa$, $D \in V[G]$ exists $C \in V$ which refines D.

Theorem

$\mathfrak{i} = \mathfrak{f} < \mathfrak{u}$ is consistent with ZFC.

Denote $\mathbf{C}_\kappa = \{ h \colon \kappa \to 2 \mid |h| < \omega \}$ ordered by reverse inclusion.
A forcing $\mathbf{P}$ is s *Cohen-preserving* if for each dense $D \subset \mathbf{C}_\kappa$, $D \in V[G]$ exists $C \in V$ which refines D.

Proposition (A. Miller)

If $\mathbf{P}$ is a proper forcing with the Sacks property, then $\mathbf{P}$ is Cohen-preserving.

Let $\mathcal{A} \subset \mathcal{P}(\omega)$ be an independent system.
For $h \in \mathbf{C}_{\mathcal{A}}$ let $\mathcal{A}^h = \bigcap \{ A^{h(A)} \mid A \in \operatorname{dom} h \}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_{\mathcal{A}}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{ h \mid h \text{ reaps } X \}$ is nonempty for each $X \subseteq \omega$.

Let $\mathcal{A} \subset \mathcal{P}(\omega)$ be an independent system.
For $h \in \mathbf{C}_{\mathcal{A}}$ let $\mathcal{A}^h = \bigcap \{ A^{h(A)} \mid A \in \operatorname{dom} h \}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_{\mathcal{A}}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{ h \mid h \text{ reaps } X \}$ is nonempty for each $X \subseteq \omega$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

Observation

A dense independent system is maximal.

Let $\mathcal{A} \subset \mathcal{P}(\omega)$ be an independent system.
For $h \in \mathbf{C}_{\mathcal{A}}$ let $\mathcal{A}^h = \bigcap \{ A^{h(A)} \mid A \in \operatorname{dom} h \}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_{\mathcal{A}}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{ h \mid h \text{ reaps } X \}$ is nonempty for each $X \subseteq \omega$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

Observation

A dense independent system is maximal.

Proposition (Goldstern–Shelah)

For each maximal independent system $\mathcal{A}$ there exists $h \in \mathbf{C}_{\mathcal{A}}$ such that $\mathcal{A} \restriction \mathcal{A}^h$ is a dense independent system.

For $h \in \mathbf{C}_\mathcal{A}$ let $\mathcal{A}^h = \bigcap\{ A^{h(A)} \mid A \in \operatorname{dom} h \}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_\mathcal{A}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{ h \mid h \text{ reaps } X \}$ is nonempty for each $X \subseteq \omega$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_\mathcal{A}$ for each $X \subseteq \omega$.

$\mathscr{F}_\mathcal{A}$ be a filter defined by $F \in \mathscr{F}_\mathcal{A}$ iff $\{ h \mid h \text{ hits } F \}$ is dense in $\mathbf{C}_\mathcal{A}$.

For $h \in \mathbf{C}_{\mathcal{A}}$ let $\mathcal{A}^h = \bigcap \{ A^{h(A)} \mid A \in \operatorname{dom} h \}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_{\mathcal{A}}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{ h \mid h \text{ reaps } X \}$ is nonempty for each $X \subseteq \omega$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{ h \mid h \text{ hits } F \}$ is dense in $\mathbf{C}_{\mathcal{A}}$.
Let $\mathscr{C}_{\mathcal{A}} = \{ \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \}$.

For $h \in \mathbf{C}_{\mathcal{A}}$ let $\mathcal{A}^h = \bigcap\{\, A^{h(A)} \mid A \in \operatorname{dom} h \,\}$.
For $X \subseteq \omega$ say that $h \in \mathbf{C}_{\mathcal{A}}$ *reaps* X if either $\mathcal{A}^h \subset^* X$ or $\mathcal{A}^h \cap X =^* \emptyset$.
If $\mathcal{A}^h \subset^* X$, say that h *hits* X.
The independent system $\mathcal{A}$ is maximal iff $\{\, h \mid h \text{ reaps } X \,\}$ is nonempty for each $X \subseteq \omega$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{\, h \mid h \text{ reaps } X \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{\, h \mid h \text{ hits } F \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$.
Let $\mathscr{C}_{\mathcal{A}} = \{\, \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \,\}$.

Lemma

An independent system $\mathcal{A}$ is dense iff $\mathcal{P}(\omega) \setminus \mathscr{F}_{\mathcal{A}}$ is generated by $\mathscr{C}_{\mathcal{A}}$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{ h \mid h \text{ hits } F \}$ is dense in $\mathbf{C}_{\mathcal{A}}$.
Let $\mathscr{C}_{\mathcal{A}} = \{ \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \}$.

Lemma

An independent system $\mathcal{A}$ is dense iff $\mathcal{P}(\omega) \setminus \mathscr{F}_{\mathcal{A}}$ is generated by $\mathscr{C}_{\mathcal{A}}$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{\, h \mid h \text{ reaps } X \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{\, h \mid h \text{ hits } F \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$. Let $\mathscr{C}_{\mathcal{A}} = \{\, \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \,\}$.

Lemma

An independent system $\mathcal{A}$ is dense iff $\mathcal{P}(\omega) \setminus \mathscr{F}_{\mathcal{A}}$ is generated by $\mathscr{C}_{\mathcal{A}}$.

Let $\mathbf{P}$ be a Cohen-preserving forcing.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{\, h \mid h \text{ reaps } X \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{\, h \mid h \text{ hits } F \,\}$ is dense in $\mathbf{C}_{\mathcal{A}}$.
Let $\mathscr{C}_{\mathcal{A}} = \{\, \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \,\}$.

Lemma

An independent system $\mathcal{A}$ is dense iff $\mathcal{P}(\omega) \setminus \mathscr{F}_{\mathcal{A}}$ is generated by $\mathscr{C}_{\mathcal{A}}$.

Let $\mathbf{P}$ be a Cohen-preserving forcing.
The set $\mathscr{C}_{\mathcal{A}}$ is absolute,
the filter $(\mathscr{F}_{\mathcal{A}})^{V[G]}$ is generated by $(\mathscr{F}_{\mathcal{A}})^{V}$.

Definition (Goldstern–Shelah)

An independent system $\mathcal{A}$ is *dense* if the set $\{ h \mid h \text{ reaps } X \}$ is dense in $\mathbf{C}_{\mathcal{A}}$ for each $X \subseteq \omega$.

$\mathscr{F}_{\mathcal{A}}$ be a filter defined by $F \in \mathscr{F}_{\mathcal{A}}$ iff $\{ h \mid h \text{ hits } F \}$ is dense in $\mathbf{C}_{\mathcal{A}}$.
Let $\mathscr{C}_{\mathcal{A}} = \{ \omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}} \}$.

Lemma

An independent system $\mathcal{A}$ is dense iff $\mathcal{P}(\omega) \setminus \mathscr{F}_{\mathcal{A}}$ is generated by $\mathscr{C}_{\mathcal{A}}$.

Let $\mathbf{P}$ be a Cohen-preserving forcing.
The set $\mathscr{C}_{\mathcal{A}}$ is absolute,
the filter $(\mathscr{F}_{\mathcal{A}})^{V[G]}$ is generated by $(\mathscr{F}_{\mathcal{A}})^{V}$.

To show that $\mathcal{A}$ is preserved as a maximal independent system in $V[G]$ it is sufficient to show that $\mathscr{C}_{\mathcal{A}}$ generates $\mathcal{P}(\omega) \setminus (\mathscr{F}_{\mathcal{A}})^{V}$ in $V[G]$.

Definition

Let B be a free sequence, $\mathcal{A}$ be a dense independent system. We say B is *associated with* $\mathcal{A}$ if B is maximal, and B generates $\mathscr{F}_{\mathcal{A}}$.

Definition

Let B be a free sequence, $\mathcal{A}$ be a dense independent system. We say B is *associated with* $\mathcal{A}$ if B is maximal, and B generates $\mathscr{F}_{\mathcal{A}}$.

Proposition

Let B be a free sequence associated with $\mathcal{A}$, $\mathbf{P}$ be a Cohen-preserving forcing, preserving $\mathcal{A}$ as a dense independent system. Then in $V[G]$ the sequence B remains to be associated with $\mathcal{A}$ (i.e. maximal).

Definition

Let B be a free sequence, $\mathcal{A}$ be a dense independent system. We say B is *associated with* $\mathcal{A}$ if B is maximal, and B generates $\mathscr{F}_{\mathcal{A}}$.

Proposition

Let B be a free sequence associated with $\mathcal{A}$, $\mathbf{P}$ be a Cohen-preserving forcing, preserving $\mathcal{A}$ as a dense independent system. Then in $V[G]$ the sequence B remains to be associated with $\mathcal{A}$ (i.e. maximal).

Proposition

Assume CH. *There exists a selective dense independent system $\mathcal{A}$ and a free sequence B associated with $\mathcal{A}$.*

Definition

Let B be a free sequence, $\mathcal{A}$ be a dense independent system. We say B is *associated with* $\mathcal{A}$ if B is maximal, and B generates $\mathscr{F}_{\mathcal{A}}$.

Proposition

Let B be a free sequence associated with $\mathcal{A}$, $\mathbf{P}$ be a Cohen-preserving forcing, preserving $\mathcal{A}$ as a dense independent system. Then in $V[G]$ the sequence B remains to be associated with $\mathcal{A}$ (i.e. maximal).

Proposition

Assume CH. *There exists a selective dense independent system $\mathcal{A}$ and a free sequence B associated with $\mathcal{A}$.*

Theorem (Shelah)

Let $\mathcal{U}$ be a non-principal ultrafilter. There exists a proper, Sacks property (i.e. Cohen-preserving) forcing $\mathbf{P}_{\mathcal{U}}$ which destroys $\mathcal{U}$ (as an ultrafilter base) and preserves selective dense independent systems.