Prikry type Forcing and True Cofinal Sequence

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Definition of $pcf(A)$

For a set of regular cardinals A and ideal I over A, we define a relation $<_I$ on $\prod A$ as follows: For $f, g \in \prod A$,

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For a filter F over A, $<_F$ is defined as $<_{F^*}$.

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Definition

For a set $A \subseteq Reg$,

$$pcf(A) = \left\{tcf\left(\prod A/I\right) \mid I \text{ is an ideal over } A \text{ and } tcf \text{ exists}\right\}$$
$$\left(= \left\{cf\left(\prod A/D\right) \mid D \text{ is an ultrafilter over } A\right\}\right)$$

$tcf(\prod A/I)$ is the least cardinality of the totally ordered cofinal subset. totally ordered cofinal subset is called "True Cofinal Sequence".

$pcf(-)$ behaves like the closure operator:

1. $A \subseteq pcf(A)$.
2. $pcf(A \cup B) = pcf(A) \cup pcf(B)$.
3. $A \subseteq B \to pcf(A) \subseteq pcf(B)$.
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Definition

$A \subseteq Reg$ is ***progressive*** *iff $min(A) > |A|$ holds.*

Theorem (Good properties of $pcf(A)$)

If $A \subseteq Reg$ is progressive, then the followings hold:

1. $pcf(A)$ *has the maximal elememt.*
2. *If A is an interval of regular cardinals, then so is* $pcf(A)$.
3. $|pcf(A)| < |A|^{+4}$
4. $pcf(pcf(A)) = pcf(A)$
5. *If $A = [\kappa, \lambda)$ and λ is a singular cardinal then* $\max pcf(A) = cf([\lambda]^\kappa, \subseteq)$.

Theorem (Shelah)

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What about $pcf(A)$ for $A \subseteq Reg$ in general?

Remark

GCH + "there is no weakly inaccessible" implies that $pcf(A)$ has good properties for every $A \subseteq Reg$.

Theorem (T.)

If κ is measurable and 2^κ is singular, then the follwoings hold:

1. $pcf(\kappa \cap Reg)$ *has no maximal element.*
2. $pcf pcf(\kappa \cap Reg) \neq pcf(\kappa \cap Reg)$.

Lemma

If κ is measurable, then $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$.

Theorem (Shelah)

If λ is singular, then there is a $A \in [\lambda \cap Reg]^{cf(\lambda)}$ *such that* $\sup A = \lambda$ *and* $tcf(\prod A / J^{bd}_{cf(\lambda)}) = \lambda^+$.

By this theorem,

$$(2^\kappa)^+ \in pcf 2^\kappa \cap Reg = pcf(pcf(\kappa \cap Reg))$$
$$(2^\kappa)^+ \notin 2^\kappa \cap Reg = pcf(\kappa \cap Reg)$$

So $pcf(pcf(\kappa \cap Reg)) \neq pcf(\kappa \cap Reg)$.

Lemma

If κ is measurable, then $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$.

Proof.

Take any $\theta \in (2^\kappa)^+ \cap Reg$.
Let $j : V \to M \simeq Ult(V, U)$.

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$$\langle f_\alpha \mid \alpha < \theta\rangle \text{ is true cofinal sequence in } \langle \prod_{\xi<\kappa} h(\xi), <_U\rangle.$$

So $\theta = tcf(\prod_{\xi<\kappa} h(\xi)/U) \in pcf(\{h(\xi) \mid \xi < \kappa\}) \subseteq pcf(\kappa \cap Reg)$.

□

In the next section, we will change a measurable cardinal κ into a singular cardinal by Prikry forcing.

Prikry forcing

Recall that Prikry forcing $\mathbb{P}_U$ by normal ultrafilter U over κ is defined by:

1. $\mathbb{P}_U = [\kappa]^{<\omega} \times U$.
2. $\langle a, A \rangle \leq \langle b, B \rangle$ iff
    1. $a \cap max(b) = b$.
    2. $A \subseteq B$.
    3. $a \setminus b \subseteq B$.

If G is $(V, \mathbb{P}_U)$-generic, then $g = \bigcup\{a \mid \exists A.\langle a, A \rangle \in G\} \in [\kappa]^\omega$ is called "Prikry sequence". We denotes κ_n as n-th ordinal in g.

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This sequence g satisfies the following properties:

1. $\forall A \in U.g \subseteq^* A$ (i.e. $\exists n < \omega\forall m \geq n.\kappa_n \in A$)
2. $\sup g = \kappa$. So $cf(\kappa) = \omega$.

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2. $\sup g = \kappa$. So $cf(\kappa) = \omega$.

And $\mathbb{P}_U$ has the following property:

Theorem (Prikry condition)

For every $a \in [\kappa]^{<\omega}$ and statement σ of forcing language, there is an $A \in U$ such that $\langle a, A\rangle$ decides σ(i.e. $\langle a, A\rangle \Vdash \sigma$ or $\langle a, A\rangle \Vdash \neg\sigma$).

Theorem (T.)

$\mathbb{P}_U \Vdash pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$

To show this, we check the following lemma:

Key Lemma

For $h \in {}^\kappa\kappa$, if $V \models [h]_U > \kappa$ is regular and $\forall \xi . h(\xi) \in Reg \wedge h(\xi) > \xi$ then,

$$\mathbb{P}_U \Vdash tcf(\prod_{n<\omega} h(\dot{\kappa_n}), <^*) = [h]_U.$$

Here $<^* = <_{J_\omega^{bd}}$, $\dot{\kappa_n}$ is a $\mathbb{P}_U$-name which denotes the n-th ordinal in Prikry sequence.

Example

$\mathbb{P}_U \Vdash tcf(\prod_{n<\omega} \dot{\kappa_n}^+, <^*) = \kappa^+.$

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Example

$\mathbb{P}_U \Vdash tcf(\prod_{n<\omega} \dot{\kappa_n}^+, <^*) = \kappa^+.$

As remark, for $f, g \in {}^\omega ON$, $f <^* g$ is defined by $\{n < \omega \mid g(n) \leq f(n)\} \in J^{bd}_\omega$. Namely,

$$\exists m < \omega \forall n \geq m . f(n) < g(n).$$

Proof from the Key Lemma

Theorem (T.)

$\mathbb{P}_U \Vdash pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$

Proof.

Let G be arbitrary $(V, \mathbb{P}_U)$-generic.
We discuss on $V[G]$, it is sufficient to show that
$(2^\kappa)^+ \cap Reg \subseteq pcf(\kappa \cap Reg)$ holds.
Pick any $\theta \in (2^\kappa)^+ \cap Reg$.
By $\kappa \cap Reg \subseteq pcf(\kappa \cap Reg)$, We may assume that $\theta > \kappa$.
Now θ is regular in V and $\theta \leq (2^\kappa)^V < j(\kappa)$. So we can take $h \in {}^\kappa\kappa$ which represents θ in M. By the key lemma
$[h]_U = \theta \in pcf(\{h(\kappa_i) \mid i < \omega\}) \subseteq pcf(\kappa \cap Reg)$. □

Proof of Key Lemma I

Let $j : V \to M$ be ultrapower defined by U. In M, for each $\alpha < [h]_U$, we can take $f_\alpha \in \prod_{\xi<\kappa} h(\xi)$ such that $[f_\alpha]_U = \alpha$.
We claim that $\mathbb{P}_U \Vdash$ "$\langle f_\alpha \restriction \dot{g} \mid \alpha < [h]_U \rangle$ is a true cofinal sequence in $\langle \prod_{n<\omega} h(\dot{\kappa_n}), <^* \rangle$". Here $f_\alpha \restriction \dot{g}$ denotes a function $n \mapsto f_\alpha(\dot{\kappa_n})$. Cleary it is in $\prod_{n<\omega} h(\dot{\kappa_n})$.
First, we check that $\mathbb{P}_U$ forces this sequence is $<^*$-increasing. To show this, let G be arbitrary $(V, \mathbb{P}_U)$-generic.

Lemma

In $V[G]$, $\alpha < \beta \to f_\alpha \restriction g <^* f_\beta \restriction g$.

Because $\alpha < \beta$, $\{\xi < \kappa \mid f_\alpha(\xi) < f_\beta(\xi)\} \in U$.
By the property of Prikry sequence,

$$\exists n < \omega \forall m \geq n.\kappa_n \in \{\xi < \kappa \mid f_\alpha(\xi) < f_\beta(\xi)\}.$$

Namely, $\exists n < \omega \forall m \geq n. f_\alpha(\kappa_n) < f_\beta(\kappa_n)$. So $f_\alpha \restriction g <^* f_\beta \restriction g$.

Proof of Key Lemma II

Next, we check that:

Lemma

$\mathbb{P}_U \Vdash \langle f \restriction \dot{g} \mid \alpha < [h]_U \rangle$ *is cofinal.*

Let $\langle a, A \rangle$ and $\dot{f} \in V^{\mathbb{P}_U}$ be such that:

$$\langle a, A \rangle \Vdash \dot{f} \in \prod_{n<\omega} h(\dot{\kappa_n}).$$

It is sufficient to prove that there is a $X \in U$ and $\gamma < \kappa$ such that:

$$\langle a, A \cap X \rangle \Vdash \dot{f} <^* f_\gamma \restriction \dot{g}.$$

For simplicity, we assume that $\langle a, A \rangle = \langle \emptyset, \kappa \rangle$.
For every $b = \{a_0, ..., a_n\} \in [\kappa]^{<\omega}$, there is a pair $\langle B_b, h_b \rangle$ satisfying that:

$$\langle b, B_b \rangle \Vdash \dot{f}(n) = h_b.$$

Proof of Key Lemma III

Note that $\langle b, Y\rangle \Vdash \dot{\kappa_n} = a_n$. By using Prikry condition, for all $\xi < h(a_n) < \kappa$, we can take B_ξ such that:

$$\langle b, B_\xi\rangle \text{ decides } \dot{f}(n) = \xi.$$

Let $B_b = \bigcap_{\xi < h(a_n)} B_\xi$. By $\langle b, B_b\rangle \Vdash \dot{f}(n) < h(\dot{\kappa_n}) = h(a_n)$ and $\langle b, B_b\rangle$ decides $\dot{f}(n) = \xi$ for every $\xi < h(a_n)$, $\langle b, B\rangle$ decides the value of $\dot{f}(n)$. So there is an $h_b < h(a_n)$ such that:

$$\langle b, B_b\rangle \Vdash \dot{f}(n) = h_b.$$

Let $H \in {}^\kappa\kappa$ be defined by:

$$H(\xi) = \sup\{h_b + 1 \mid b \in [\kappa]^{<\omega} \wedge \max b = \xi\}.$$

By the regularity of $h(\xi)$, $H(\xi) < h(\xi)$.

Proof of Key Lemma IV

By the definition of H, for every $b = \{a_0, ..., a_n\} \in [\kappa]^{<\omega}$,

$$\langle b, B_b \rangle \Vdash \dot{f}(n) = h_b < H(\max b) = H(\dot{\kappa_n}).$$

In M, $[H]_U < [h]_U$. So there is a γ such that $[H]_U < [f_\gamma]_U$. Let $X = \{\xi < \kappa \mid H(\xi) < f_\gamma(\xi)\} \cap \triangle_b B_b \in U$. Here $\triangle_b B_b = \{\xi < \kappa \mid \forall b \in [\kappa]^{<\omega}. \max b < \xi \to \xi \in B_b\}$.
This X witnesses that $\langle \emptyset, X \rangle \Vdash \dot{f} <^* f_\gamma \restriction \dot{g}$.
Pick any $n < \omega$. Let $\langle b, B \rangle \leq \langle \emptyset, X \rangle$ be taken as arbitrary extension. We may assume that $|b| > n$.
Now b has a form $\{a_0, ..., a_n, ..., a_m\}$, $\langle b, B \rangle$ and $\langle \{a_0, ..., a_n\}, B_{\{a_0, ..., a_n\}} \rangle$ has a common extension which forces $\dot{f}(n) < f_\gamma(\dot{\kappa_n})$.
So we get

$$\langle \emptyset, X \rangle \Vdash \dot{f} <^* f_\gamma \restriction \dot{g}$$

. □

Corollary

If κ is measurable and 2^κ is singular, then $\mathbb{P}_U$ forces that $pcf(\kappa \cap Reg)$ has no maximal element and $pcf(pcf(\kappa \cap Reg)) \neq pcf(\kappa \cap Reg)$.

So pcf operation does not always behave like the closure operator.
Moreover, we got the following theorem:

Theorem (T.)

If $\kappa = \kappa_0 < \cdots < \kappa_{n-1}$ are supercompact cardinals, then there is a poset which forces that:

1. $pcf(\kappa \cap Reg) \subsetneq pcf(pcf(\kappa \cap Reg)) \cdots \subsetneq pcf^{n+1}(\kappa \cap Reg)$.
2. $cf(\kappa) = \omega$.

On uncountable cofinality

Magidor's changing cofinality forcing also preserves $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$.

Theorem (Magidor)

For a measurable cardinal κ and regular cardinal $\lambda < \kappa$, if there is a increasing sequence (w.r.t. Mitchell order) of ultrafilters $\langle U_\alpha \mid \alpha < \lambda \rangle$ over κ then there is a poset $\mathbb{M}$ forces that:

1. *κ is strong limit singular of $cf(\kappa) = \lambda$.*
2. *$\mathcal{P}^V(\lambda) = \mathcal{P}(\lambda)$.*

Theorem (T.)

$\mathbb{M} \Vdash pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$.

Question

For a large cardinal κ, $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$ is preserved by some Prikry type forcing(e.g. usual Prikry forcing, Gitik-Sharon's diagonal Prikry type forcing, Magidor forcing).

Question

Is it consistent that $pcf(\kappa \cap Reg)$ is not interval of regular cardinals for singular κ?

Remark

If strong limit singular κ is a non-fixed point of $\aleph$ function(i.e. $\kappa = \aleph_\lambda > \lambda$), then $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$.

Question

For a large cardinal κ and small set $A \subseteq 2^\kappa \cap Reg$, is there Prikry type forcing which forces $pcf(\kappa \cap Reg) \cap A = \emptyset$?

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It is consistent that κ is inaccessible and $\kappa^{++} \in pcf(\kappa \cap Reg)$ but $\kappa^+ \notin pcf(\kappa \cap Reg)$.

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Question

For other large cardinal(e.g. weakly compact) κ, does $pcf(\kappa \cap Reg) = (2^\kappa)^+ \cap Reg$ hold? What about $\kappa^+ \in pcf \kappa \cap Reg$

Question

For singular cardinal κ, $pp(\kappa)$ is the supremum of the set of all $tcf(\prod A/I)$ where $A \in [\kappa]^{cf(\kappa)}$, $\sup A = \kappa$ and $J^{bd}_{cf(\kappa)} \subseteq I$ is an ideal over $cf(\kappa)$.
We say that $pp(\kappa)$ is pseudo power of κ.

Theorem (Shelah)

If κ is a singular cardinal which is non-fixed point of $\aleph$-function and $2^{cf(\kappa)} < \kappa$, then $pp(\kappa) = \kappa^{cf(\kappa)}$

Question

Is $pp(\kappa) \neq \kappa^{cf(\kappa)}$ for some singular κ consistent with ZFC?