On generalized notion of higher stationarity

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Section 1

Higher stationary sets of ordinals

n-stationary subsets of ordinals

Definition (n-stationary subsets of ordinals)

By induction on $n < \omega$, define the notion of n-stationary subsets of $\kappa \in \mathrm{On}$ as follows:

- $S \subseteq \kappa$ is 0-stationary in κ if S is unbouned in κ.
- S is n-stationary in κ if for all $m < n$ and all m-stationary $T \subseteq \kappa$ there is $\mu \in S$ s.t. $T \cap \mu$ is m-stationary in μ.
- κ is n-stationary if κ is n-stationary in κ.

- S is 1-stationary in κ iff S is stationary in κ.
- S is 2-stationary in κ iff every stationary subset of κ reflects to some $\mu \in S$. In particular, κ is 2-stationary iff the stationary reflection in κ holds.
- This notion of n-stationary sets is relevant to the proof theory:
    - topological semantics of provability logic. (Beklemishev et al.)
    - ordinal analysis of the theory ZFC + Π^1_n-Indescribable Card. Axiom. (Arai)

Family of non-n-stationary sets

Definition

For $\kappa \in \mathrm{On}$ and $n < \omega$, let

$$\mathrm{NS}^n_\kappa \;:=\; \{S \subseteq \kappa \mid S \text{ is not } n\text{-stationary in } \kappa\}.$$

- NS^1_κ is the non-stationary ideal over κ.
- For $n \geq 2$, NS^n_κ may not be an ideal:
  Suppose every stationary subset of κ reflects, but there are stationary $T_0, T_1 \subseteq \kappa$ which do not reflect simultaneously. Let

  $$S_i \;:=\; \{\mu < \kappa \mid T_i \cap \mu \text{ is not stationary in } \mu\}.$$

  Then $S_0, S_1 \in \mathrm{NS}^2_\kappa$, but $S_0 \cup S_1 = \kappa \notin \mathrm{NS}^2_\kappa$.

Π^1_n-indescribable sets

Definition (Π^1_n-indescribability)

Suppose $\kappa \in \mathrm{On}$ and $n < \omega$.

- $S \subseteq \kappa$ is Π^1_n-indescribable in κ if for all $P \subseteq V_\kappa$ and all Π^1_n-sentence φ with $(V_\kappa, \in, P) \models \varphi$, there is $\mu \in S$ with $(V_\mu, \in, P \cap V_\mu) \models \varphi$.
- κ is Π^1_n-indescribable if κ is Π^1_n-indescribable in κ.
- $\mathrm{NI}^n_\kappa \ := \ \{S \subseteq \kappa \mid S \text{ is not } \Pi^1_n\text{-indescribable in } \kappa\}$.

Fact ((1),(2),(4):Lévy, (3):Scott)

1. κ is Π^1_0-indescribable iff κ is inaccessible.
2. For an inaccessible cardinal κ, $S \subseteq \kappa$ is Π^1_0-indescribable in κ iff S is stationary in κ.
3. κ is Π^1_1-indescribable iff κ is weakly compact.
4. NI^n_κ is a normal ideal over κ.

Π^1_1-indescribability and 2-stationarity

The following is easy:

Fact

If S is Π^1_1-indescribable in κ, then S is 2-stationary in κ.

In L, the converse also holds:

Theorem (Jensen)

Assume $V = L$. Let κ be a regular uncountable cardinal.
If S is 2-stationary in κ, then S is Π^1_1-indescribable in κ.

Kunen proved that the 2-stationarity does not imply the Π^1_1-indescribability in general. In fact, the consistency strengths are different:

Theorem (Shelah-Mekler)

The consistency strength of the existence of a 2-stationary cardinal is strictly weaker than that of a Π^1_1-indescribable cardinal.

Preservation of 2-stationarity and continuum

Theorem (Shelah)

Suppose $\kappa \in \mathrm{On}$. Then every c.c.c. forcing preserves 2-stationary subsets of κ.

Corollary

It is consistent that there is $\kappa \leq 2^\omega$ which is 2-stationary.

Π^1_n-indescribability and $n+1$-stationarity

Fact

If S is Π^1_n-indescribable in κ, then S is $n+1$-stationary in κ.

Theorem (Bagaria-Magidor-S.)

Assume $V = L$. Let κ be a regular uncountable cardinal.
If S is $n+1$-stationary in κ, then S is Π^1_n-indescribable in κ.

Theorem (Bagaria-Magidor-Mancilla)

For $n \in \omega \setminus \{0\}$, the consistency strength of the existence of an $n+1$-stationary cardinal is strictly weaker than that of a Π^1_n-indescribable cardinal.

Preservation of n-stationarity and continuum

We have the following preservation theorem for n-stationary sets:

Theorem

Assume GCH. Suppose $n \in \omega$, κ is a regular uncountable cardinal and $\rho < \kappa$. Assume NS^m_μ is a normal ideal over μ for all regular $\mu \leq \kappa$ and all m with $1 \leq m \leq n$. Then every ρ-c.c. forcing preserves n-stationary subsets of κ.

Note that, in L, $\mathrm{NS}^m_\mu = \mathrm{NI}^{m-1}_\mu$ is a normal ideal. So the assumption of Theorem holds in L. Thus we have the following corollary by a forcing over L:

Corollary

It is consistent that there is a cardinal $\kappa \leq 2^\omega$ which is n-stationary for all $n < \omega$.

Outline of Proof of Theorem

By induction on n, we prove that for all regular μ with $\rho < \mu \leq \kappa$ and all ρ-c.c. poset $\mathbb{P}$, we have the following in $V^{\mathbb{P}}$:

$$\mathrm{NS}^n_\mu \; = \; \overline{(\mathrm{NS}^n_\mu)^V}\,.$$

[Proof of "$\subseteq$" (i.e. $S \notin \overline{(\mathrm{NS}^n_\mu)^V} \;\Rightarrow\; S$ is n-stationary)]

- We may assume $|\mathbb{P}| \leq \mu$. Suppose $\mathbb{P} = \mu$. We work in V.
- Suppose $m < n$ and $\dot{T}$ is a $\mathbb{P}$-name for an m-stationary subset of μ.
- It suffices to prove that the following C is in the dual filter F of $(\mathrm{NS}^n_\mu)^V$:

  $$C \;:=\; \{\nu < \mu \mid \Vdash_{\mathbb{P}} \dot{T} \cap \nu \text{ is } m\text{-stationary}\}.$$

- For each $p \in \mathbb{P}$, the following T_p is m-stationary in μ:

  $$T_p \;:=\; \{\alpha < \mu \mid \exists q \leq p,\; q \Vdash_{\mathbb{P}} \alpha \in \dot{T}\}.$$

- By the normality of $(\mathrm{NS}^n_\mu)^V$,

  $$D \;:=\; \{\nu < \mu \mid \forall p < \nu,\; T_p \cap \nu \text{ is } m\text{-stat. \& } \mathbb{P} \cap \nu \subseteq_{\mathrm{c}} \mathbb{P}\} \;\in\; F.$$

- Moreover, $D \subseteq C$. □

Section 2

Higher stationary sets in $\mathcal{P}_\kappa(\lambda)$

n-stationary subsets of $\mathcal{P}_\kappa(A)$

Definition (n-stationary subsets of $\mathcal{P}_\kappa(A)$)

For a regular cardinal κ, a set $A \supseteq \kappa$ and $n < \omega$:

- $S \subseteq \mathcal{P}_\kappa(A)$ is 0-stationary in $\mathcal{P}_\kappa(A)$ if S is $\subseteq$-cofinal in $\mathcal{P}_\kappa(A)$.
- $S \subseteq \mathcal{P}_\kappa(A)$ is n-stationary in $\mathcal{P}_\kappa(A)$ if for all $m < n$ and all m-stationary $T \subseteq \mathcal{P}_\kappa(A)$, there is $B \in S$ s.t.
  - $\mu := B \cap \kappa$ is a regular cardinal,
  - $T \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$.
- $\mathcal{P}_\kappa(A)$ is n-stationary if $\mathcal{P}_\kappa(A)$ is n-stationary in $\mathcal{P}_\kappa(A)$.
- $\mathrm{NS}^n_{\kappa,A} \;:=\; \{S \subseteq \mathcal{P}_\kappa(A) \mid S \text{ is not } n\text{-stationary in } \mathcal{P}_\kappa(A)\}$.

- If $\mathcal{P}_\kappa(A)$ is 1-stationary, then κ is weakly Mahlo.
- Suppose κ is Mahlo. Then $\mathrm{NS}^1_{\kappa,A}$ is the smallest strongly normal ideal over $\mathcal{P}_\kappa(A)$. If $|\mathcal{P}_\kappa(A)| = |A|$ and $f : \mathcal{P}_\kappa(A) \to A$ is a bijection, then $\mathrm{NS}^1_{\kappa,A} = \mathrm{NS}_{\kappa,A} \restriction S$, where

$$S \;=\; \{x \in \mathcal{P}_\kappa(\lambda) \mid \mu := x \cap \kappa \in \mathrm{Reg} \;\&\; f[\mathcal{P}_\mu(x)] \subseteq x\}\,.$$

Π^1_n-indescribability in $\mathcal{P}_\kappa(A)$

For a regular κ and a set $A \supseteq \kappa$, we define $V_\alpha(\kappa, A)$ by induction on α.

- $V_0(\kappa, A) := A$,
- $V_{\alpha+1}(\kappa, A) := \mathcal{P}_\kappa(V_\alpha(\kappa, A)) \cup V_\alpha(\kappa, A)$,
- $V_\alpha(\kappa, A) := \bigcup_{\beta<\alpha} V_\beta(\kappa, A)$ for a limit α.

Definition (Baumgartner)

Suppose κ is a regular cardinal, $A \supseteq \kappa$ and $n < \omega$.

- $S \subseteq \mathcal{P}_\kappa(A)$ is Π^1_n-indescribable in $\mathcal{P}_\kappa(A)$ if for all $P \subseteq V_\kappa(\kappa, A)$ and all Π^1_n-sentence φ with $(V_\kappa(\kappa, A), \in, P) \models \varphi$, there is $B \in S$ such that
  - $\mu := B \cap \kappa$ is a regular cardinal,
  - $(V_\mu(\mu, B), \in, P \cap V_\mu(\mu, B)) \models \varphi$.
- $\mathcal{P}_\kappa(A)$ is Π^1_n-indescribable if $\mathcal{P}_\kappa(A)$ is Π^1_n-indescribable in $\mathcal{P}_\kappa(A)$.
- $\mathrm{NI}^n_{\kappa,A} := \{S \subseteq \mathcal{P}_\kappa(A) \mid S \text{ is not } \Pi^1_n\text{-indescribable in } \mathcal{P}_\kappa(A)\}$.

Theorem (Abe, Car)

1. $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ is Π^1_1-indescribable. $\Rightarrow$ κ is λ-supercompact.
   $\Rightarrow$ $\mathcal{P}_\kappa(\lambda)$ is Π^1_n-indescribable for all $n \in \omega$.
2. $\mathrm{NI}^n_{\kappa,A}$ is a strongly normal ideal over $\mathcal{P}_\kappa(A)$.

Π^1_n-indescribability and $n+1$-stationarity

Proposition

- If S is Π^1_n-indescribable in $\mathcal{P}_\kappa(\lambda)$, then S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$.
- Suppose κ is Mahlo. Then S is Π^1_0-indescribable in $\mathcal{P}_\kappa(\lambda)$ iff S is 1-stationary in $\mathcal{P}_\kappa(\lambda)$.

Recall that, in L, S is Π^1_n-indescribable in κ iff S is $n+1$-stationary in κ.
We do not know whether its analogy is consistent in a non-trivial way:

Question

Is the following consistent?

- For all regular κ, all $\lambda \geq \kappa$, all $S \subseteq \mathcal{P}_\kappa(\lambda)$ and all $n < \omega$,
  S is Π^1_n-indescribable in $\mathcal{P}_\kappa(\lambda)$ iff S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$.
- There is a supercompact cardinal.

Strongly compact cardinal

Recall that if κ is λ-supercompact, then $\mathcal{P}_\kappa(\lambda)$ is n-stationary for all $n < \omega$.

Question

Is $\mathcal{P}_\kappa(\lambda)$ n-stationary for all $n < \omega$ if κ is λ-strongly compact ?

We can prove that the strong compactness of κ does not imply the stationary reflection in $\mathcal{P}_\kappa(\lambda)$:

For all stationary $S \subseteq \mathcal{P}_\kappa(\lambda)$, there is $B \in \mathcal{P}_\kappa(\lambda)$ s.t.

- *$\mu := B \cap \kappa$ is a regular cardinal,*
- *$S \cap \mathcal{P}_\mu(B)$ is stationary in $\mathcal{P}_\mu(B)$.*

Proposition

It is consistent that there is a strongly compact cardinal κ s.t. the stationary reflection in $\mathcal{P}_\kappa(\kappa^+)$ fails.

But, we do not know the answer of the above question.

Preservation of n-stationary sets under forcing

We have the following preservation theorem:

Theorem

Assume GCH. Suppose $n < \omega$, κ is a Mahlo cardinal and $\rho < \kappa \leq \lambda$. Assume $\mathrm{NS}^{m}_{\mu,\nu}$ is a strongly normal ideal over $\mathcal{P}_{\mu}(\nu)$ for all m with $1 \leq m \leq n$ and all μ, ν with $\mu \leq \kappa$ and $\nu \leq \lambda$. Then every ρ-c.c. forcing preserves n-stationary subsets of $\mathcal{P}_{\kappa}(\lambda)$.

But, we do not know the assumption of Theorem is consistent...

One of difficulties to deal with $\mathcal{P}_{\kappa}(\lambda)$ in forcing extensions is that $\mathcal{P}_{\kappa}(\lambda)$ changes after a forcing. But, in our context, this is not problematic:

Lemma

Assume GCH. Suppose κ is a Mahlo cardinal and $\lambda \geq \kappa$. Let W be a ρ-c.c. forcing extension of V for some $\rho < \kappa$. Then $\mathcal{P}_{\kappa}(\lambda)^{W} \setminus \mathcal{P}_{\kappa}(\lambda)^{V}$ is not 1-stationary in W.

This is almost immediate from the fact that every ρ-c.c. forcing extension has the ρ^{+}-approximation property.

n-stationarity and continuum

It is not hard to prove the following:

Proposition

It is consistent that there is a cardinal $\kappa \leq 2^\omega$ such that $\mathcal{P}_\kappa(\lambda)$ is 2-stationary for all $\lambda \geq \kappa$.

But, we do not know whether this can be generalized to n-stationarity:

Question

For $n \geq 3$, is it conssistent that there is a cardinal $\kappa \leq 2^\omega$ such that $\mathcal{P}_\kappa(\lambda)$ is n-stationary for all $\lambda \geq \kappa$?